

# TANGLE FLOER HOMOLOGY AND COBORDISMS BETWEEN TANGLES

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ABSTRACT. Fix a commutative algebra  $\mathbb{A}$  over  $\mathbb{Z}/2\mathbb{Z}$ .  $\mathbb{A}$ -tangles are introduced as refinements of balanced sutured manifolds, and form the objects of the category  $\mathbb{A}\text{-Tangles}$ .  $\mathbb{A}$ -cobordisms between  $\mathbb{A}$ -tangles are also introduced as the morphisms of  $\mathbb{A}\text{-Tangles}$ . Denote the category of  $\mathbb{A}$ -modules and  $\mathbb{A}$ -homomorphisms between them by  $\mathbb{A}\text{-Modules}$ . Associated with every  $\mathbb{A}$ -module  $\mathbb{M}$  we construct a functor

$$\mathrm{HF}^{\mathbb{M}} : \mathbb{A}\text{-Tangles} \longrightarrow \mathbb{A}\text{-Modules},$$

called the *tangle Floer homology functor*, which associates the  $\mathbb{A}$ -module  $\mathrm{HF}^{\mathbb{M}}(\mathcal{T})$  to the  $\mathbb{A}$ -tangle  $\mathcal{T}$ , and assigns an  $\mathbb{A}$ -homomorphism

$$\mathfrak{f}_{\mathcal{C}}^{\mathbb{M}} : \mathrm{HF}^{\mathbb{M}}(\mathcal{T}) \rightarrow \mathrm{HF}^{\mathbb{M}}(\mathcal{T}')$$

to every  $\mathbb{A}$ -cobordism  $\mathcal{C}$  from  $\mathcal{T}$  to  $\mathcal{T}'$ .

Applying the above machinery to decorated cobordisms between links, we get functorial maps on link Floer homology. As a byproduct, we construct an invariant  $u'(K)$  for every knot  $K \subset S^3$  which gives a lower bound on the unknotting number of  $K$ , while it is always greater than or equal to  $\tau(K)$ . Moreover,  $u'(K)$  is positive if  $K$  is non-trivial.

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## 1. INTRODUCTION

**1.1. Introduction and background.** Ozsváth and Szabó introduced Heegaard Floer homology for closed three dimensional manifolds [OS2, OS3] which resulted in powerful tools for the study of various structures in low dimensional topology, including invariants for knots [OS1, Ras1, Ef1], for links [OS5], for contact structures [OS6] and for sutured manifolds [Ju3, AE]. Juhász and Thurston [JT] showed that the Heegaard Floer groups associated with three-dimensional objects (closed manifolds, links and sutured manifolds) are in fact functors which associate a concrete module to any of the aforementioned topological objects, rather than just the isomorphism class of it. Typically, Heegaard Floer homology groups come in different flavours, which are denoted by  $\widehat{\text{HF}}$ ,  $\text{HF}^+$ ,  $\text{HF}^-$  and  $\text{HF}^\infty$ , besides many other flavours which appear in knot and link Floer homology theories. The authors gave a framework that brings all flavours of Heegaard Floer homology under the same roof in [AE], giving an extension of the construction of Juhász [Ju3].

**Definition 1.1.** A tangle  $(M, T)$  is a compact oriented 3-manifold  $M$  without closed components and with a fixed decomposition of its boundary as  $\partial M = \partial^+ M \sqcup \partial^- M$ , together with a properly embedded oriented one-manifold  $T = \coprod_{i=1}^n T_i$  with no closed components in  $M$ . Moreover,  $\partial^- T_\circ \subset \partial^- M$  and  $\partial^+ T_\circ \subset \partial^+ M$ , giving the maps (induced by the above inclusions)

$$i_T^\circ : \pi_0(T) \rightarrow \pi_0(\partial^\circ M), \quad \circ = +, -.$$

The tangle  $(M, T)$  is called balanced if  $i_T^+$  and  $i_T^-$  are both surjective and for every connected component  $M_\circ$  of  $M$ ,  $\chi(\partial^+ M_\circ) = \chi(\partial^- M_\circ)$ .

Every tangle determines a sutured manifold without toroidal components, and conversely, every sutured manifold without toroidal components determines a tangle. The set  $\text{Spin}^c(M)$  of  $\text{Spin}^c$  structures over a tangle  $(M, T)$  is the set of homology classes of nonzero vector fields on  $M$  which restrict to the outward normal of  $\partial^+ M$  and the inward normal of  $\partial^- M$ .

Fix an algebra  $\mathbb{A}$  over  $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$  (which will always be commutative through this paper). Given  $\mathbf{u} : \pi_0(T) \rightarrow \mathbb{A}$ , for every  $s \in \pi_0(\partial^\circ M)$  define

$$\mathbf{u}(s) := \prod_{\substack{t \in \pi_0(T) \\ i_T^\circ(t) = s}} \mathbf{u}(t) \in \mathbb{A}.$$

**Definition 1.2.** An  $\mathbb{A}$ -tangle is a 4-tuple  $\mathcal{T} = [M, T, \mathfrak{s}, \mathbf{u} : \pi_0(T) \rightarrow \mathbb{A}]$  where  $(M, T)$  is a balanced tangle,  $\mathfrak{s} \in \text{Spin}^c(M)$  is a  $\text{Spin}^c$  structure over  $M$  and the followings are satisfied.

- (1) If  $s \in \pi_0(\partial M)$  corresponds to a connected component of  $\partial M$  with positive genus then  $\mathbf{u}(s) = 0$ .
- (2) For every connected component  $M_\circ$  of  $M$

$$\sum_{s \in \pi_0(\partial^+ M_\circ)} \mathbf{u}(s) = \sum_{s \in \pi_0(\partial^- M_\circ)} \mathbf{u}(s)$$

For instance, we may take  $\mathbb{A} = \mathbb{F}$  and let  $\mathbf{u}$  map everything to zero. The corresponding  $\mathbb{F}$ -tangles are then precisely the balanced tangles which are equipped by  $\text{Spin}^c$  structures.

If  $\mathcal{T}$  is an  $\mathbb{A}$ -tangle, we use  $[M_{\mathcal{T}}, T_{\mathcal{T}}, \mathfrak{s}_{\mathcal{T}}, \mathbf{u}_{\mathcal{T}}]$  to denote the corresponding 4-tuple. Given an  $\mathbb{A}$ -module  $\mathbb{M}$  and associated with an  $\mathbb{A}$ -tangle  $\mathcal{T}$  the construction of authors in [AE] defines a Floer homology group

$$\text{HF}^{\mathbb{M}}(\mathcal{T}) = H_*(\text{CF}(\mathcal{T}) \otimes_{\mathbb{A}} \mathbb{M})$$

so that its isomorphism type is an invariant of  $\mathcal{T}$ .

In light of Juhász and Thurston's naturality discussions in [JT], one can in fact strengthen the above result as follows. Let  $\mathbb{A}\text{-Tang}$  denote the category of  $\mathbb{A}$ -tangles. The morphisms from an  $\mathbb{A}$ -tangle  $\mathcal{T} = [M, T, \mathfrak{s}, \mathbf{u}]$  to another

$\mathbb{A}$ -tangle  $\mathcal{T}' = [M', T', \mathfrak{s}', \mathfrak{u}']$  are the diffeomorphisms  $d : (M, T) \rightarrow (M', T')$  such that  $d^*\mathfrak{s}' = \mathfrak{s}$  and the following diagram is commutative.

$$\begin{array}{ccc} \pi_0(T) & \xrightarrow{d_*} & \pi_0(T') \\ & \searrow \mathfrak{u} & \swarrow \mathfrak{u}' \\ & \mathbb{A} & \end{array}$$

Here  $d^* : \text{Spin}^c(M') \rightarrow \text{Spin}^c(M)$  and  $d_* : \pi_0(T) \rightarrow \pi_0(T')$  are the maps induced by the diffeomorphism  $d$ . Let  $\mathbb{A}\text{-Mod}$  denote the category of  $\mathbb{A}$ -modules together with the isomorphisms between them. Using Theorem 2.39 from [JT] one can in fact prove the following theorem (see Section 3).

**Theorem 1.3.** *For every algebra  $\mathbb{A}$  over  $\mathbb{F}$  and every  $\mathbb{A}$ -module  $\mathbb{M}$ , assigning the  $\mathbb{A}$ -module  $\text{HF}^{\mathbb{M}}(\mathcal{T})$  to the  $\mathbb{A}$ -tangle  $\mathcal{T}$  in  $\mathbb{A}\text{-Tang}$  gives a functor*

$$\text{HF}^{\mathbb{M}} : \mathbb{A}\text{-Tang} \longrightarrow \mathbb{A}\text{-Mod}.$$

**1.2. Main results.** Associated with a cobordism  $W$  from a closed oriented 3-manifold  $M$  to another closed 3-manifold  $M'$ , and a  $\text{Spin}^c$  class  $\mathfrak{t} \in \text{Spin}^c(W)$ , Ozsváth and Szabó construct a homomorphism

$$\mathfrak{f}_{W, \mathfrak{t}}^{\circ} : \text{HF}^{\circ}(M, \mathfrak{t}|_M) \longrightarrow \text{HF}^{\circ}(M', \mathfrak{t}|_{M'}).$$

With Theorem 1.3 in place, it is natural to ask if the construction of Ozsváth and Szabó may be extended to an invariant for cobordisms between tangles. We first need to define the notion of cobordisms between tangles.

**Definition 1.4.** *A cobordism  $(W, F)$  from  $(M, T)$  to  $(M', T')$  consists of a smooth oriented four-manifold  $W$ , with boundary and corners and without closed components, and a properly embedded smooth oriented surface  $F$  in  $W$ , with boundary and corners and without closed components, such that:*

- (1) *The boundary  $(\partial W, \partial F)$  of  $(W, F)$  consists of a horizontal part*

$$\begin{aligned} (\partial_h W, \partial_h F) &= (\partial_h^+ W, \partial_h^+ F) \sqcup (\partial_h^- W, \partial_h^- F) \\ &= (\partial^+ M \times I, \partial^+ T \times I) \sqcup (\partial^- M \times I, \partial^- T \times I) \\ &= (\partial^+ M' \times I, \partial^+ T' \times I) \sqcup (\partial^- M' \times I, \partial^- T' \times I) \end{aligned}$$

*and a vertical part  $(\partial_v W, \partial_v F) = -(M, T) \sqcup (M', T')$ , with corners*

$$(\partial_v W, \partial_v F) \cap (\partial_h W, \partial_h F) = (\partial M, \partial T) \sqcup (\partial M', \partial T').$$

- (2) *For every component  $F_{\circ}$  of  $F$  the orientation induced on  $\partial F_{\circ}$  by the orientation of  $F_{\circ}$  agrees with the orientation inherited from  $-T \sqcup T'$ .*

*The cobordism  $(W, F)$  is called stable if  $(M, T)$  and  $(M', T')$  are balanced and for every connected component  $F_{\circ}$  of  $F$  which is not homeomorphic to a disk,  $T \cap F_{\circ}$  and  $T' \cap F_{\circ}$  have more than one connected component. The inclusions of  $T$  and  $T'$  in  $F$  give the maps*

$$\mathfrak{J}_T : \pi_0(T) \rightarrow \pi_0(F) \quad \text{and} \quad \mathfrak{J}_{T'} : \pi_0(T') \rightarrow \pi_0(F).$$

Let us assume that  $(W, F)$  is a stable cobordism as above. Associated with this  $(W, M \cup M')$  we obtain the cohomology long exact sequence

$$\dots \longrightarrow H^2(W, M \cup M', \mathbb{Z}) \xrightarrow{\iota} H^2(W, \mathbb{Z}) \xrightarrow{\pi} H^2(M \cup M', \mathbb{Z}) \longrightarrow \dots$$

One may then consider the space  $\text{Spin}^c(W)$  of  $\text{Spin}^c$  structures on  $W$ , as discussed in Subsection 2.2. If  $\mathfrak{t}, \mathfrak{t}' \in \text{Spin}^c(W)$  satisfy  $\mathfrak{t}|_M = \mathfrak{t}'|_M = \mathfrak{s}$  and  $\mathfrak{t}|_{M'} = \mathfrak{t}'|_{M'} = \mathfrak{s}'$  then  $\mathfrak{t} - \mathfrak{t}' \in \text{Ker}(\pi) = \text{Im}(\iota)$ . By an *affine set* of  $\text{Spin}^c$  structures over  $W$  we mean a subset  $\mathfrak{T} = \mathfrak{t} + H_{\mathfrak{T}} \subset \text{Spin}^c(W)$  which is determined by a  $\text{Spin}^c$  structure  $\mathfrak{t} \in \text{Spin}^c(W)$  and a submodule of  $\text{Ker}(\pi) \subset H^2(W, \mathbb{Z})$ . In particular, if this fixed submodule is trivial, then  $\mathfrak{T}$  consists of a single  $\text{Spin}^c$  structure  $\mathfrak{t}$ . If  $\mathfrak{T}$  is an affine set of  $\text{Spin}^c$  structures on  $W$  then  $\mathfrak{T}|_M$  and  $\mathfrak{T}|_{M'}$  are well-defined.

**Definition 1.5.** An  $\mathbb{A}$ -cobordism  $\mathcal{C} = [W, F, \mathfrak{T}, \mathfrak{u}]$  from  $\mathcal{T}$  to  $\mathcal{T}'$  consists of a stable cobordism  $(W, F)$  from  $(M, T)$  to  $(M', T')$ , an affine set  $\mathfrak{T} \subset \text{Spin}^c(W)$  of  $\text{Spin}^c$  structure over  $W$ , and a map  $\mathfrak{u} : \pi_0(F) \rightarrow \mathbb{A}$  so that

$$\mathcal{T} = [M, T, \mathfrak{s} = \mathfrak{T}|_M, \mathfrak{u} \circ j_T] \quad \text{and} \quad \mathcal{T}' = [M', T', \mathfrak{s}' = \mathfrak{T}|_{M'}, \mathfrak{u} \circ j_{T'}]$$

are  $\mathbb{A}$ -tangles. If  $\mathcal{C}$  is an  $\mathbb{A}$ -cobordism from  $\mathcal{T}$  to  $\mathcal{T}'$ , we write  $\mathcal{C} : \mathcal{T} \rightsquigarrow \mathcal{T}'$ .

We use  $[W_{\mathcal{C}}, F_{\mathcal{C}}, \mathfrak{T}_{\mathcal{C}}, \mathfrak{u}_{\mathcal{C}}]$  to denote the 4-tuple associated with an  $\mathbb{A}$ -cobordism  $\mathcal{C}$ . For every  $\mathbb{A}$ -cobordism  $\mathcal{C}$ , we set  $\text{Spin}^c(\mathcal{C}) = \text{Spin}^c(W_{\mathcal{C}})$ . When  $\mathfrak{T}$  consists of a single  $\text{Spin}^c$  structure, we will use the notation  $\mathfrak{t}$  (or similar notation). This assumption will be made from Section 2 through Subsection 7.3.

Allowing affine sets of  $\text{Spin}^c$  structures instead of single  $\text{Spin}^c$  structures over cobordisms allows us to compose  $\mathbb{A}$ -cobordisms. In fact, if

$$\mathcal{C}_1 = [W_1, F_1, \mathfrak{T}_1, \mathfrak{u}_1] : \mathcal{T} \rightsquigarrow \mathcal{T}' \quad \text{and} \quad \mathcal{C}_2 = [W_2, F_2, \mathfrak{T}_2, \mathfrak{u}_2] : \mathcal{T}' \rightsquigarrow \mathcal{T}''$$

are  $\mathbb{A}$ -cobordisms, then the cohomology exact sequence

$$\dots \longrightarrow H^1(M_{\mathcal{T}'}, \mathbb{Z}) \xrightarrow{\delta} H^2(W, \mathbb{Z}) \xrightarrow{\pi} H^2(W_1, \mathbb{Z}) \oplus H^2(W_2, \mathbb{Z}) \longrightarrow \dots$$

and the submodules corresponding to  $W_1$  and  $W_2$  determine a submodule of  $H^2(W, \mathbb{Z})$  as their pre-image under  $\pi$ . Since  $\mathfrak{T}_1$  restricts to  $\mathfrak{s} = \mathfrak{s}_{\mathcal{T}}$  on  $M = M_{\mathcal{T}}$  and  $\mathfrak{T}_2$  restricts to  $\mathfrak{s}'' = \mathfrak{s}_{\mathcal{T}''}$  on  $M'' = M_{\mathcal{T}''}$ , this submodule determines an affine set  $\mathfrak{T}$  of  $\text{Spin}^c$  structures on  $W$ , which is bigger than  $\mathfrak{T}_1 \times \mathfrak{T}_2$  unless the map  $\delta$  in the above sequence is trivial. We may then compose the  $\mathbb{A}$ -cobordisms  $\mathcal{C}_1$  and  $\mathcal{C}_2$  to obtain

$$\mathcal{C} = \mathcal{C}_1 \cup_{\mathcal{T}'} \mathcal{C}_2 : \mathcal{T} \rightsquigarrow \mathcal{T}''.$$

From Definition 1.5, we obtain a category  $\mathbb{A}\text{-Tangles}$ . The objects of  $\mathbb{A}\text{-Tangles}$  are  $\mathbb{A}$ -tangles and the morphism are  $\mathbb{A}$ -cobordisms. The category  $\mathbb{A}\text{-Tang}$  is a subcategory of  $\mathbb{A}\text{-Tangles}$ : given a diffeomorphism

$$d : [M, T, \mathfrak{s}, \mathfrak{u}] \rightarrow [M', T', \mathfrak{s}', \mathfrak{u}']$$

define an  $\mathbb{A}$ -cobordism  $\mathcal{C} = [W, F, \mathfrak{t}, \mathfrak{u}_F]$ , where  $(W, F) = (M, T) \times [0, 1]$ ,  $(M, T) \times \{1\}$  is parametrized by

$$d : (M, T) \times \{1\} \rightarrow (M', T'),$$

and the  $\text{Spin}^c$  structure  $\mathfrak{t}$  and the map  $\mathfrak{u}_F$  are trivially determined by  $\mathfrak{s}$  and  $\mathfrak{u}$ , respectively. Let  $\mathbb{A}\text{-Modules}$  denote the category of  $\mathbb{A}$ -modules, where the morphisms are given by  $\mathbb{A}$ -homomorphisms between  $\mathbb{A}$ -modules. Once again,  $\mathbb{A}\text{-Mod}$  is a subcategory of  $\mathbb{A}\text{-Modules}$ . The main theorem of this paper is then the following.

**Theorem 1.6.** *Suppose that the  $\mathbb{F}$ -algebra  $\mathbb{A}$  is given. For every  $\mathbb{A}$ -module  $\mathbb{M}$ , Heegaard Floer homology gives a functor*

$$\text{HF}^{\mathbb{M}} : \mathbb{A}\text{-Tangles} \longrightarrow \mathbb{A}\text{-Modules}.$$

Moreover, the restriction of  $\text{HF}^{\mathbb{M}}$  to the subcategory  $\mathbb{A}\text{-Tang}$  is the functor  $\text{HF}^{\mathbb{M}}$  from Theorem 1.3.

We denote the  $\mathbb{A}$ -homomorphism  $\text{HF}^{\mathbb{M}}(\mathcal{C})$  associated with the  $\mathbb{A}$ -cobordism  $\mathcal{C} : \mathcal{T} \rightsquigarrow \mathcal{T}'$  by

$$\mathfrak{f}_{\mathcal{C}}^{\mathbb{M}} : \text{HF}^{\mathbb{M}}(\mathcal{T}) \rightarrow \text{HF}^{\mathbb{M}}(\mathcal{T}').$$

If  $\phi : \mathbb{M} \rightarrow \mathbb{M}'$  is a homomorphism of  $\mathbb{A}$ -modules, we obtain a corresponding homomorphism of  $\mathbb{A}$ -modules

$$\mathfrak{f}^{\phi} : \text{HF}^{\mathbb{M}}(\mathcal{T}) \longrightarrow \text{HF}^{\mathbb{M}'}(\mathcal{T}).$$

Given an  $\mathbb{A}$ -cobordism  $\mathcal{C} : \mathcal{T} \rightsquigarrow \mathcal{T}'$ , the following diagram is commutative:

$$\begin{array}{ccc} \text{HF}^{\mathbb{M}}(\mathcal{T}) & \xrightarrow{\mathfrak{f}_{\mathcal{C}}^{\mathbb{M}}} & \text{HF}^{\mathbb{M}}(\mathcal{T}') \\ \mathfrak{f}^{\phi} \downarrow & & \downarrow \mathfrak{f}^{\phi} \\ \text{HF}^{\mathbb{M}'}(\mathcal{T}) & \xrightarrow{\mathfrak{f}_{\mathcal{C}}^{\mathbb{M}'}} & \text{HF}^{\mathbb{M}'}(\mathcal{T}'). \end{array}$$

Moreover, given a short exact sequence

$$0 \longrightarrow \mathbb{M} \xrightarrow{\iota} \mathbb{M}' \xrightarrow{\pi} \mathbb{M}'' \longrightarrow 0$$

of  $\mathbb{A}$ -modules, we obtain a corresponding exact triangle of tangle Floer homology  $\mathbb{A}$ -modules:

$$\begin{array}{ccc} \text{HF}^{\mathbb{M}}(\mathcal{T}) & \xleftarrow{\mathfrak{f}^{\delta}} & \text{HF}^{\mathbb{M}''}(\mathcal{T}) \\ & \searrow \mathfrak{f}^{\iota} & \nearrow \mathfrak{f}^{\pi} \\ & \text{HF}^{\mathbb{M}'}(\mathcal{T}) & \end{array}$$

where  $\mathfrak{f}^{\delta}$  is a connecting homomorphism.

**Remark 1.7.** *A similar construction, in the context of pointed links and cobordisms between them, is independently given by Ian Zemke in [Zem].*

**1.3. Examples and applications.** Let us first review some of the familiar cases of the above construction.

**Example 1.** Every oriented pointed 3-manifold  $\mathcal{Y} = (Y, p)$  (with  $p \in Y$ ) may be regarded as a balanced sutured manifold, and thus a balanced tangle: Remove a ball neighborhood of  $p$  and put a single suture on the sphere boundary of the resulting manifold. Equivalently, one may consider a small arc passing through  $p$  and remove a pair of balls around the end-points of this arc to obtain an associated tangle. Let us denote the resulting tangle by  $(M_{\mathcal{Y}}, T_{\mathcal{Y}})$ . Set  $\mathbb{A} = \mathbb{F}[\mathbf{u}]$  and define  $\mathbf{u}_{\mathcal{Y}} : \pi_0(T_{\mathcal{Y}}) \rightarrow \mathbb{A}$  by sending  $T_{\mathcal{Y}}$  to  $\mathbf{u}$ . For every  $\mathfrak{s} \in \text{Spin}^c(Y)$ ,  $\mathcal{T}_{\mathcal{Y}, \mathfrak{s}} = [M_{\mathcal{Y}}, T_{\mathcal{Y}}, \mathfrak{s}, \mathbf{u}_{\mathcal{Y}}]$  is an  $\mathbb{A}$ -tangle. If  $X$  is a smooth 4-manifold with  $\partial X = -Y \amalg Y'$ , where  $Y$  and  $Y'$  are connected oriented 3-manifolds, and if  $\sigma$  is a simple path in  $X$  from  $p \in Y$  to  $p' \in Y'$ , then  $\mathcal{X} = (X, \sigma)$  gives a cobordism  $(W_{\mathcal{X}}, F_{\mathcal{X}})$  from  $(M_{\mathcal{Y}}, T_{\mathcal{Y}})$  to  $(M_{\mathcal{Y}'}, T_{\mathcal{Y}'})$ , where  $F_{\mathcal{X}}$  is a disk. Once again, sending  $F_{\mathcal{X}}$  to  $\mathbf{u} \in \mathbb{A}$  defines a map  $\mathbf{u}_{\mathcal{X}} : \pi_0(F_{\mathcal{X}}) \rightarrow \mathbb{A}$  and  $\mathcal{C}_{\mathcal{X}, \mathfrak{t}} = [W_{\mathcal{X}}, F_{\mathcal{X}}, \mathfrak{t}, \mathbf{u}_{\mathcal{X}}]$  is an  $\mathbb{A}$ -cobordism from  $\mathcal{T}_{\mathcal{Y}, \mathfrak{t}_Y}$  to  $\mathcal{T}_{\mathcal{Y}', \mathfrak{t}_{Y'}}$  for every  $\mathfrak{t} \in \text{Spin}^c(X)$ . Choosing  $\mathbb{M}$  equal to  $\mathbb{F}, \mathbb{F}[\mathbf{u}], \mathbb{F}[\frac{1}{\mathbf{u}}]$  or  $\mathbb{F}[\mathbf{u}, \frac{1}{\mathbf{u}}]$  gives the Heegaard Floer homology group  $\text{HF}^{\mathbb{M}}(\mathcal{T}_{\mathcal{Y}, \mathfrak{s}})$ , which is equal to  $\widehat{\text{HF}}(Y, \mathfrak{s}; \mathbb{F}), \text{HF}^-(Y, \mathfrak{s}; \mathbb{F}), \text{HF}^+(Y, \mathfrak{s}; \mathbb{F})$  and  $\text{HF}^{\infty}(Y, \mathfrak{s}; \mathbb{F})$ , respectively. The  $\mathbb{A}$ -homomorphism  $\mathfrak{f}_{\mathcal{X}, \mathfrak{t}}^{\mathbb{M}} = \mathfrak{f}_{\mathcal{C}_{\mathcal{X}, \mathfrak{t}}}^{\mathbb{M}}$  is then the cobordism map of Ozsváth and Szabó in any of the aforementioned cases.

**Example 2.** Suppose that  $L$  is an oriented link in a connected oriented closed 3-manifold  $Y$  and  $\mathbf{p} = \{p_1, \dots, p_n\}$  is a collection of markings on  $L$ , so that each component of  $L$  contains at least one marked point. We may consider a collection of  $n$  small arcs on  $L$  containing  $\mathbf{p}$ , and remove small balls from a neighborhood of the endpoints of these arcs. This gives a 3-manifold  $M_{L, \mathbf{p}}$  with  $2n$  sphere boundary components. The orientation on  $L$  may be used to decompose this boundary into  $n$  spheres in  $\partial^+ M$  and  $n$  spheres in  $\partial^- M$ . After changing the orientation of the small arcs, we also obtain a tangle  $T_{L, \mathbf{p}}$  with  $2n$  connected components which connect the negative boundary to the positive boundary. Let  $\mathbb{A} = \mathbb{F}[\mathbf{u}_1, \dots, \mathbf{u}_n, \mathbf{v}]$ , and map the small arc corresponding to  $p_i$  to the variable  $\mathbf{u}_i$ . Map the remaining  $n$  arcs to  $\mathbf{v}$ . Associated with the corresponding representation  $\mathbf{u}_{L, \mathbf{p}} : \pi_0(T_{L, \mathbf{p}}) \rightarrow \mathbb{A}$  and any  $\mathfrak{s} \in \text{Spin}^c(Y)$  we obtain an  $\mathbb{A}$ -tangle  $\mathcal{T}_{L, \mathbf{p}, \mathfrak{s}} = [M_{L, \mathbf{p}}, T_{L, \mathbf{p}}, \mathfrak{s}, \mathbf{u}_{L, \mathbf{p}}]$ . Correspondingly, we define the homology groups

$$\text{HF}^{\mathbb{M}}(Y, L, \mathbf{p}, \mathfrak{s}) = \text{HF}^{\mathbb{M}}(\mathcal{T}_{L, \mathbf{p}, \mathfrak{s}}),$$

which give the usual link Floer homology groups associated with links using different  $\mathbb{A}$ -modules  $\mathbb{M}$ .

Suppose that  $(L, \mathbf{p}) \subset Y$  and  $(L', \mathbf{p}') \subset Y'$  are marked links as above. For a decorated cobordism  $\mathcal{Z} = (Z, F, \sigma)$  consisting of a smooth oriented 4-dimensional cobordism  $Z$  from  $Y$  to  $Y'$ , a properly embedded smooth oriented surface  $F \subset Z$  which connects  $L$  to  $L'$  and a properly embedded oriented 1-manifold  $\sigma \subset F$  which connects  $\mathbf{p}$  to  $\mathbf{p}'$  i.e.  $\partial^- \sigma = \mathbf{p}$  and  $\partial^+ \sigma = \mathbf{p}'$ , we construct a cobordism  $(W_{\mathcal{Z}}, F_{\mathcal{Z}})$  connecting  $(M_{L, \mathbf{p}}, T_{L, \mathbf{p}})$  to  $(M_{L', \mathbf{p}'}, T_{L', \mathbf{p}'})$ . We require that any connected component  $F - \sigma$  with positive genus intersect  $L$  and  $L'$  in more than one connected component, to achieve stability. The representation  $\mathbf{u}_{L, \mathbf{p}}$  considered earlier extends to a map  $\mathbf{u}_{\mathcal{Z}} : \pi_0(F_{\mathcal{Z}}) \rightarrow \mathbb{A}$ . Thus, associated with every  $\text{Spin}^c$  structure  $\mathbf{t} \in \text{Spin}^c(Z)$  we obtain an  $\mathbb{A}$ -cobordism

$$\mathcal{C}_{\mathcal{Z}, \mathbf{t}} = [W_{\mathcal{Z}}, F_{\mathcal{Z}}, \mathbf{t}, \mathbf{u}_{\mathcal{Z}}] : \mathcal{T}_{L, \mathbf{p}, \mathbf{t}|_Y} \leadsto \mathcal{T}_{L', \mathbf{p}', \mathbf{t}|_{Y'}}.$$

Correspondingly, we obtain the cobordism maps

$$\mathfrak{f}_{\mathcal{Z}, \mathbf{t}}^{\mathbb{M}} : \text{HF}^{\mathbb{M}}(Y, L, \mathbf{p}, \mathbf{t}|_Y) \rightarrow \text{HF}^{\mathbb{M}}(Y', L', \mathbf{p}', \mathbf{t}|_{Y'}).$$

The functoriality of link Floer homology then follows from our main theorem.

As a by-product of the above construction, we find a way to bound the unknotting number  $u(K)$  of a knot  $K$  from below. If  $K \subset S^3$  is a knot and  $p$  is an arbitrary point on  $K$ ,  $\mathcal{T}_{K, p}$  is an  $\mathbb{A}$ -tangle, where  $\mathbb{A} = \mathbb{F}[\mathbf{u}, \mathbf{v}]$ . The chain complex  $\text{CF}(K, p)$  is then a module over  $\mathbb{A}$ , and may be decomposed as

$$\text{CF}(K, p) = \bigoplus_{s \in \mathbb{Z}} \text{CF}(K, p, s),$$

where  $\text{CF}(K, p, s)$  is generated over  $\mathbb{F}$  by the elements of the form  $\mathbf{u}^a \mathbf{v}^b \mathbf{x}$  with  $\underline{s}(\mathbf{x}) + b - a = s$  and  $\mathbf{x}$  a generator of the Heegaard Floer chain complex. Similarly, we set  $\mathbb{A}(s) = \langle \mathbf{u}^a \mathbf{v}^b \mid a + b = s \rangle_{\mathbb{F}}$ . We may then talk about the chain maps  $\text{CF}(K, p) \rightarrow \mathbb{A}$  and  $\mathbb{A} \rightarrow \text{CF}(K, p)$  which are homogeneous of some fixed degree.

**Definition 1.8.** For a knot  $K \subset S^3$ , let  $u'(K)$  denote the least integer  $m$  with the property that there are homogenous chain map

$$\mathfrak{f} : \text{CF}(K, p) \rightarrow \mathbb{A} \quad \text{and} \quad \mathfrak{g} : \mathbb{A} \rightarrow \text{CF}(K, p)$$

of non-negative degree such that the  $\mathfrak{f} \circ \mathfrak{g} = \mathbf{v}^m$  and

$$\mathfrak{g} \circ \mathfrak{f} : \text{CF}(K) \rightarrow \text{CF}(K)$$

is chain homotopy equivalent to multiplication by  $\mathbf{v}^m$ .

We prove the following theorem in the last section.

**Theorem 1.9.** If  $K$  is a knot in  $S^3$ , then  $u'(K)$  is a lower bound for the unknotting number  $u(K)$ . Moreover,  $u'(K) \geq |\tau(K)|$  where  $\tau(K)$  is the  $\tau$ -invariant of Ozváth and Szabó.



Unlike some of the other bounds from Heegaard Floer theory for the unknotting number of knots in  $S^3$  which are also lower bounds for the slice genus,  $u'(K)$  does not bound the slice genus from below. In fact, if  $K$  is non-trivial then  $u'(K)$  is positive.

**1.4. Outline of the paper.** The paper is organized as follows. In Section 2 we review Heegaard poly-tuples and  $\text{Spin}^c$  structures over them. Then in Section 3 we follow the footsteps of Juhász and Thurston [JT] to show that tangle Floer homology for  $\mathbb{A}$ -tangles gives functors from  $\mathbb{A}\text{-Tang}$  to  $\mathbb{A}\text{-Mod}$ .

In Section 4 we study parametrized Cerf decompositions of cobordisms between tangles. We show that any two parametrized Cerf decompositions for a stable cobordism can be connected by a sequence of Cerf moves.

In Section 5 we define cobordism maps for parametrized cobordisms associated with attaching one or three handles and show that the map is invariant. In Section 6 we introduce a special  $\mathbb{A}$ -tangle  $\mathcal{T}_F$  associated to the positive boundary of any  $\mathbb{A}$ -cobordism together with a distinguished generator  $\Theta_F \in \text{HF}(\mathcal{T}_F)$ . The distinguished generator makes it possible to define invariant cobordism maps for cobordisms parametrized with framed links and framed arcs.

In Section 7 we define cobordism maps for arbitrary  $\mathbb{A}$ -cobordisms by composing cobordism maps constructed in Sections 5 and 6 for cobordisms parametrized by framed 0-spheres, framed knots and arcs and framed 2-spheres. We prove that this map is in fact an invariant and does not depend on the parametrized Cerf decomposition. Moreover, we show that this construction gives functors from  $\mathbb{A}\text{-Tangles}$  to  $\mathbb{A}\text{-Modules}$ .

Finally in Section 8 we discuss some special cases and applications. In particular, decorated cobordisms between pointed links induce functorial maps on link Floer homology.

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## 2. TANGLES, $\text{Spin}^c$ STRUCTURES AND HEEGAARD POLY-TUPLES

**2.1. Tangles and sutured manifolds.** Sutured manifolds were introduced by Gabai in [Gab1, Gab2, Gab3]. Throughout this paper, we use a less general family of sutured manifolds by excluding toroidal sutures.

**Definition 2.1.** A sutured manifold  $(X, \tau)$  is a compact oriented 3-manifold  $X$  with boundary, together with a set of pairwise disjoint oriented simple closed curves  $\tau = \{\tau_1, \dots, \tau_\kappa\}$  on  $\partial X$ . We will denote by  $A(\tau_i)$  a tubular neighborhood of  $\tau_i$  in  $\partial X$ , which will be an annulus. We let

$$A(\tau) = A(\tau_1) \cup \dots \cup A(\tau_\kappa).$$

Every connected component of  $\mathfrak{R}(\tau) = \partial X - A(\tau)^\circ$  is oriented with the orientation induced from  $X$ , where  $A(\tau)^\circ$  denotes the interior of  $A(\tau)$ . Furthermore, we require that  $\mathfrak{R}(\tau) = \mathfrak{R}^+(\tau) \sqcup \mathfrak{R}^-(\tau)$  where  $\mathfrak{R}^+(\tau)$  (respectively,  $\mathfrak{R}^-(\tau)$ ) denotes the union of components of  $\mathfrak{R}(\tau)$  such that the orientation induced on  $\tau$  as the boundary of  $\mathfrak{R}^+(\tau)$  (respectively,  $\mathfrak{R}^-(\tau)$ ) agrees with (respectively, is the opposite of) the orientation of  $\tau$ .

Every sutured manifold  $(X, \tau)$  determines a tangle  $(M, T)$ , where  $M = \overline{X}$  is obtained from  $X$  by filling the sutures (i.e. attaching 2-handles along the sutures) and  $T$  is the set of cocores of these 2-handles. The orientation on  $\tau$  induces an orientation on  $T$  and the decomposition  $\partial \mathfrak{R}(\tau) = \mathfrak{R}^+(\tau) \sqcup \mathfrak{R}^-(\tau)$  induces a decomposition  $\partial M = \partial^+ M \sqcup \partial^- M$  of the boundary of  $M$ . Conversely, every tangle  $(M, T)$  determines a sutured manifold.

Note that a tangle  $(M, T)$  is balanced if and only if the corresponding sutured manifold is balanced in the sense of [Ju3, Definition 2.2].

Given a commutative algebra  $\mathbb{A}$  over  $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$  and  $\mathbf{u} : \pi_0(T) \rightarrow \mathbb{A}$ , for every  $s \in \pi_0(\partial M)$  we may define

$$\mathbf{u}(s) := \prod_{\substack{t \in \pi_0(T) \\ \iota_T^\circ(t) = s}} \mathbf{u}(t) \in \mathbb{A}$$

where  $\circ \in \{+, -\}$  and  $\iota_T^\circ : \pi_0(T) \rightarrow \pi_0(\partial^\circ M)$  is the map induced by inclusion. Following Definition 1.2, an  $\mathbb{A}$ -tangle is a 4-tuple  $[M, T, \mathfrak{s}, \mathbf{u} : \pi_0(T) \rightarrow \mathbb{A}]$  where  $(M, T)$  is a balanced tangle and  $\mathfrak{s} \in \text{Spin}^c(M)$  is a  $\text{Spin}^c$  structure on  $M$ . Moreover, if  $s \in \pi_0(\partial M)$  corresponds to a connected component of  $\partial M$  with positive genus then  $\mathbf{u}(s) = 0$ , and for every connected component  $M_\circ$  of  $M$

$$\sum_{s \in \pi_0(\partial^+ M_\circ)} \mathbf{u}(s) = \sum_{s \in \pi_0(\partial^- M_\circ)} \mathbf{u}(s).$$

For instance, we may take  $\mathbb{A} = \mathbb{F}$ . Then any balanced tangle which is equipped with a  $\text{Spin}^c$  structure, together with the map  $\mathbf{u}$  which maps everything to zero, is an  $\mathbb{A}$ -tangle.

In [AE], we introduced a  $\mathbb{Z}$ -algebra  $\mathbb{A}_\tau$  associated to the boundary of any balanced sutured manifold  $(X, \tau)$ . Assume  $\tau = \sqcup_{i=1}^\kappa \tau_i$ ,

$$\mathfrak{R}^-(\tau) = \bigcup_{i=1}^k R_i^- \quad \text{and} \quad \mathfrak{R}^+(\tau) = \bigcup_{j=1}^l R_j^+.$$

Associated with the connected components of  $\mathfrak{R}(\tau)$ , consider the elements

$$\mathbf{u}_i^- := \prod_{\tau_j \subset \partial R_i^-} \mathbf{u}_j, \quad i = 1, \dots, k \quad \text{and} \quad \mathbf{u}_i^+ := \prod_{\tau_j \subset \partial R_i^+} \mathbf{u}_j, \quad i = 1, \dots, l,$$

in the free  $\mathbb{Z}$ -algebra  $\mathbb{Z}[\mathbf{u}_1, \dots, \mathbf{u}_\kappa]$  generated by  $\mathbf{u}_1, \dots, \mathbf{u}_\kappa$ . Then

$$\mathbb{A}_\tau = \frac{\mathbb{Z}[\mathbf{u}_1, \dots, \mathbf{u}_\kappa]}{\langle \mathbf{u}^+(\tau) - \mathbf{u}^-(\tau) \rangle + \langle \mathbf{u}_i^+ \mid g_i^+ > 0 \rangle + \langle \mathbf{u}_i^- \mid g_i^- > 0 \rangle}$$

where  $\mathbf{u}^-(\tau) = \sum_{i=1}^k \mathbf{u}_i^-$ ,  $\mathbf{u}^+(\tau) = \sum_{i=1}^l \mathbf{u}_i^+$  and  $g_i^\bullet$  denotes the genus of  $R_i^\bullet$  for  $\bullet = +, -$ .

For a connected balanced tangle  $(M, T)$  (i.e. a balanced tangle with  $M$  connected) let  $\mathbb{A}_T$  denote the algebra associated to the corresponding balanced sutured manifold and  $\mathbf{u}_T : \pi_0(T) \rightarrow \mathbb{A}_T$  denote the map which sends the component  $T_i$  of  $T$  to the corresponding variable  $\mathbf{u}_i \in \mathbb{A}_T$ . Then, for every  $\text{Spin}^c$  structure  $\mathfrak{s} \in \text{Spin}^c(M)$ , the balanced tangle  $(M, T)$  together with  $\mathfrak{s}$  and a map  $\mathbf{u} : \pi_0(T) \rightarrow \mathbb{A}$  becomes an  $\mathbb{A}$ -tangle if and only if  $\mathbf{u} = \phi \circ \mathbf{u}_T$  for a homomorphism  $\phi$  from  $\mathbb{A}_T$  to  $\mathbb{A}$ . Hence, given an  $\mathbb{A}$ -tangle  $\mathcal{T}$ , our construction in [AE] defines a chain complex  $\text{CF}(\mathcal{T})$  whose chain homotopy type is an invariant of  $\mathcal{T}$ . Furthermore, associated with any  $\mathbb{A}$ -module  $\mathbb{M}$  the isomorphism type of the Floer homology group

$$\text{HF}^\mathbb{M}(\mathcal{T}) = H_*(\text{CF}(\mathcal{T}) \otimes_{\mathbb{A}} \mathbb{M})$$

is an invariant of  $\mathcal{T}$ .

Following the notation set in Definition 1.2 and Definition 1.5 we may define a pair of categories associated with every commutative algebra  $\mathbb{A}$  over  $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ .

**Definition 2.2.** *The category of  $\mathbb{A}$ -tangles, denoted by  $\mathbb{A}\text{-Tang}$ , is given as follows. Its objects are  $\mathbb{A}$ -tangles and the set of morphisms from an  $\mathbb{A}$ -tangle  $\mathcal{T} = [M, T, \mathfrak{s}, \mathbf{u}_T]$  to another  $\mathbb{A}$ -tangle  $\mathcal{T}' = [M', T', \mathfrak{s}', \mathbf{u}_{T'}]$  are diffeomorphisms  $d : (M, T) \rightarrow (M', T')$  such that  $d^*\mathfrak{s}' = \mathfrak{s}$  and the diagram*

$$\begin{array}{ccc} \pi_0(T) & \xrightarrow{d} & \pi_0(T') \\ & \searrow \mathbf{u}_T & \swarrow \mathbf{u}_{T'} \\ & \mathbb{A} & \end{array}$$

is commutative. Every such diffeomorphism is denoted by  $d : \mathcal{T} \rightarrow \mathcal{T}'$ . The cobordism category of  $\mathbb{A}$ -tangles,  $\mathbb{A}\text{-Tangles}$ , is the category whose objects are  $\mathbb{A}$ -tangles, while its set of morphisms consists of  $\mathbb{A}$ -cobordisms.

The category  $\mathbb{A}\text{-Tang}$  is a subcategory of  $\mathbb{A}\text{-Tangles}$ : given a diffeomorphism  $d : \mathcal{T} \rightarrow \mathcal{T}'$ , we have an  $\mathbb{A}$ -cobordism  $[W, F, \mathfrak{t}, \mathfrak{u}]$  where

$$(W, F) = (M_{\mathcal{T}}, T_{\mathcal{T}}) \times [0, 1],$$

$\mathfrak{t}$  is the trivial extension of  $\mathfrak{s}_{\mathcal{T}}$  to  $W$  and  $(M_{\mathcal{T}}, T_{\mathcal{T}}) \times \{1\}$  is identified with  $(M_{\mathcal{T}'}, T_{\mathcal{T}'})$  via  $d$ . Moreover,  $\mathfrak{u} : \pi_0(F) \rightarrow \mathbb{A}$  is the map induced by  $\mathfrak{u}_{\mathcal{T}}$ . When  $d$  is the identity, we denote the above  $\mathbb{A}$ -cobordism by  $\mathcal{T} \times [0, 1]$ .

**2.2.  $\text{Spin}^c$  structures over cobordisms.** Given a stable cobordism  $(W, F)$  from  $(M, T)$  to  $(M', T')$ , let  $\xi$  denote the oriented 2-plane field in  $\partial_h W = \partial M \times [0, 1]$  consisting of the tangent planes of the surfaces  $\partial M \times \{t\}$  for any  $t \in [0, 1]$ . Fix an almost complex structure  $J_0$  on  $TW|_{\partial_h W}$  such that  $\xi$  consists of complex lines i.e. 2-planes in  $\xi$  are invariant under  $J_0$ . One may further extend  $J_0$  to an almost complex structure on  $F$  by requiring that  $J_0$  preserves the tangent space of  $F$ . Note that the set of almost complex structures  $J_0$  with the above property is contractible.

**Definition 2.3.** A  $\text{Spin}^c$  structure on  $W$  is the homology class of a pair  $(J, P)$ , where

- (1)  $P \subset W - \partial_h W$  is a finite collection of points,
- (2)  $J$  is an almost complex structure on  $W - P$  with  $J|_{\partial_h W} = J_0$ .

We call the pairs  $(J_1, P_1)$  and  $(J_2, P_2)$  homologous if there exists a compact 1-manifold  $C \subset W - \partial_h W$  without closed components with  $\partial C = P_1 \cup P_2$  such that  $J_1|_{W-C}$  and  $J_2|_{W-C}$  are isotopic relative to  $\partial_h W$ . Denote the set of  $\text{Spin}^c$  structures on  $W$  by  $\text{Spin}^c(W)$ . Similarly, a relative  $\text{Spin}^c$  structure on the pair  $(W, F)$  is the homology class of a pair  $(J, P)$ , where  $P \subset W - (\partial_h W \cup F)$  is a finite collection of points and  $J$  is an almost complex structure on  $W - P$  which agrees with  $J_0$  over  $\partial_h W \cup F$ . The notion of homologous pairs is defined similarly. The set of relative  $\text{Spin}^c$  structures over  $(W, F)$  is denoted by  $\text{Spin}^c(W, F)$ .

Since  $J_0$  is chosen from a contractible family, the above definition does not depend on the particular choice of  $J_0$ .

After fixing a metric over the 4-manifold  $W$ , any oriented 2-plane field which extends  $\xi$  and is defined in the complement  $W - P$  determines a corresponding almost complex structure. It is not hard to show that  $\text{Spin}^c(W)$  is an affine space over  $H^2(W, \partial_h W; \mathbb{Z})$ .

For every  $\text{Spin}^c$  structure  $\mathfrak{s}$  over  $W$ , if  $J$  is an almost complex structure (defined in the complement of a finite set of interior points of  $W$ ) representing  $\mathfrak{s}$ , the plane field  $\{V_p\}_{p \in M \sqcup M'}$  defined as

$$V_p = T_p(\partial_v W) \cap J(T_p(\partial_v W))$$

is invariant under the almost complex structure  $J$  and gives the corresponding  $\text{Spin}^c$  structures  $\mathfrak{s}|_M$  and  $\mathfrak{s}|_{M'}$ .

**2.3. Heegaard poly-tuples.** Let  $H = (\Sigma, \alpha^1, \dots, \alpha^m, \mathbf{z})$  be a balanced Heegaard diagram. This means that  $\Sigma$  is a closed oriented surface and for any  $1 \leq i \leq m$ ,  $\alpha^i = \{\alpha_1^i, \dots, \alpha_\ell^i\}$  is a set of  $\ell$  disjoint simple closed curves on  $\Sigma$  for some  $\ell > 0$ . Moreover,  $\mathbf{z} = \{z_1, \dots, z_\kappa\}$  is a set of marked points in  $\Sigma - \alpha^1 - \alpha^2 - \dots - \alpha^m$  such that for any  $1 \leq i \leq m$  every connected component of  $\Sigma - \alpha^i$  intersects  $\mathbf{z}$ .

Associated with the balanced Heegaard diagram  $H$ , we define a pair  $(W_H, F_H)$  as follows. For every  $1 \leq i \leq m$ , the Heegaard diagram  $(\Sigma, \alpha^i, \emptyset, \mathbf{z})$  determines a tangle  $(U_i, T_i)$ , where  $U_i = C[\alpha^i]$  is the compression body determined by  $\alpha^i$  i.e. it is obtained from  $\Sigma \times [0, 1]$  by attaching 2-handles along the curves  $\alpha^i \times \{1\}$ , and  $T_i = \mathbf{z} \times [0, 1]$ . Thus  $\partial U_i = \partial^- U_i \sqcup \partial^+ U_i$ , where

$$\partial^- U_i = \Sigma \quad \text{and} \quad \partial^+ U_i = \Sigma[i]$$

and  $\Sigma[i]$  is obtained by cutting  $\Sigma$  along  $\alpha_i$  and attaching disks to the boundary components of the resulting surface. We denote the finite set  $T_i \cap \partial^- U_i = \mathbf{z} \times \{0\}$  by  $\mathbf{z}_i$ .

Let  $\mathbb{D}_m$  be a  $m$ -gon with the vertices  $v_1, \dots, v_m$ , labelled in clockwise order, and edges  $e_1, \dots, e_m$ , where  $e_i$  connects  $v_{i-1}$  to  $v_i$  for  $i = 2, \dots, m$  and  $e_1$  connects  $v_m$  to  $v_1$ . Define

$$W_H := \frac{(\Sigma \times \mathbb{D}_m) \sqcup (\coprod_{i=1}^m U_i \times e_i)}{\Sigma \times e_i \sim \partial^- U_i \times e_i} \quad \text{and} \\ F_H := \frac{(\mathbf{z} \times \mathbb{D}_m) \sqcup (\coprod_{i=1}^m T_i \times e_i)}{\mathbf{z} \times e_i \sim \mathbf{z}_i \times e_i}.$$

We smooth the corners of  $W_H$  and  $F_H$  along  $v_i \times \Sigma$  for  $1 \leq i \leq m$ . Corresponding to any vertex  $v_i$ , we obtain a balanced tangle in  $(\partial W_H, \partial F_H)$  determined by the Heegaard diagram  $(\Sigma, \alpha^i, \alpha^{i+1}, \mathbf{z})$  for  $1 \leq i \leq m-1$ , denoted by  $(M_{i,i+1}, T_{i,i+1})$ , and by the Heegaard diagram  $(\Sigma, \alpha^m, \alpha^1, \mathbf{z})$  for  $i = m$ , denoted by  $(M_{m,1}, T_{m,1})$ . Let

$$M' := M_{1,2} \sqcup M_{2,3} \sqcup \dots \sqcup M_{m,1} \subset \partial W_H \quad \text{and} \quad Z := \partial W_H - \text{int}(M')$$

Thus,  $Z$  is a product and

$$Z = (\partial^- M_{1,2} \sqcup \dots \sqcup \partial^- M_{m,1}) \times [0, 1].$$

Fix an almost complex structure  $J_0$  on  $TW_H|_Z$  such that for any  $t \in [0, 1]$  the tangent planes to the surface  $(\partial^- M_{1,2} \sqcup \dots \sqcup \partial^- M_{m,1}) \times \{t\}$  are complex lines. The almost complex  $J_0$  may further be extended to  $F_H$  so that it preserves the tangent space of  $F_H$ , i.e. the tangent planes of  $F_H$  are all complex lines. We may then talk about the  $\text{Spin}^c$  structures on  $W_H$  and the relative  $\text{Spin}^c$  structures on  $(W_H, F_H)$ .

**Definition 2.4.** *The set of  $\text{Spin}^c$  structures on  $W_H$ , denoted by  $\text{Spin}^c(W_H)$  is defined as the set of homology classes of the pairs  $(J, P)$  consisting of a finite set of points  $P$  in the interior of  $W_H$  and an almost complex structure  $J$  on  $W_H - P$  such that  $J|_Z = J_0$ .*

*Similarly, the set of relative  $\text{Spin}^c$  structures on  $(W_H, F_H)$ , denoted by  $\text{Spin}^c(W_H, F_H)$  is defined as the set of homology classes of the pairs  $(J, P)$  consisting of a finite set  $P$  of points in the interior of  $W_H - F_H$  and an almost complex structure  $J$  on  $W_H - P$  such that  $J|_{Z \cup F_H} = J_0$ .*

As before, it is not hard to show that  $\text{Spin}^c(W_H)$  is an affine space over the cohomology group  $H^2(W_H, \mathbb{Z}; \mathbb{Z})$ , while  $\text{Spin}^c(W_H, F_H)$  is an affine space over the cohomology group  $H^2(W_H, Z \cup F_H; \mathbb{Z})$ .

Let  $\mathbb{T}_i \subset \text{Sym}^\ell(\Sigma)$  denote the torus  $\alpha_1^i \times \dots \times \alpha_\ell^i$  and fix

$$\mathbf{x}_i \in \mathbb{T}_i \cap \mathbb{T}_{i+1}, \quad i = 1, \dots, m-1 \quad \text{and} \quad \mathbf{x}_m \in \mathbb{T}_m \cap \mathbb{T}_1.$$

Let  $\pi_2(\mathbf{x}_1, \dots, \mathbf{x}_m)$  denote the set of homotopy classes of the maps

$$\Psi : \mathbb{D}_m \rightarrow \text{Sym}^\ell(\Sigma) \quad \text{s.t.} \quad \Psi(v_i) = \mathbf{x}_i, \quad \Psi(e_i) \subset \mathbb{T}_i, \quad i = 1, \dots, m.$$

If  $\pi_2(\mathbf{x}_1, \dots, \mathbf{x}_m)$  is non-empty then Proposition 3.3 from [GW] implies that there is an affine correspondence

$$\pi_2(\mathbf{x}_1, \dots, \mathbf{x}_m) \simeq \text{Ker} \left( \bigoplus_{i=1}^m \text{Span}(\alpha^i) \rightarrow H_2(\Sigma; \mathbb{Z}) \right) \simeq H^2(W_H, \partial W_H; \mathbb{Z})$$

where  $\text{Span}(\alpha^i)$  denotes the submodule of  $H_1(\Sigma; \mathbb{Z})$  spanned by the elements of  $\alpha^i$ .

The following proposition is a re-statement of Proposition 3.7 from [GW]:

**Proposition 2.5.** *With the above notation fixed, there is a well-defined map*

$$\mathfrak{s}_H : \pi_2(\mathbf{x}_1, \dots, \mathbf{x}_m) \rightarrow \text{Spin}^c(W_H)$$

*such that for every  $\Psi \in \pi_2(\mathbf{x}_1, \dots, \mathbf{x}_m)$*

$$\mathfrak{s}_H(\Psi)|_{M_{i,i+1}} = \mathfrak{s}(\mathbf{x}_i) \quad \text{for } i = 1, \dots, m-1 \quad \text{and} \quad \mathfrak{s}_H(\Psi)|_{M_{m,1}} = \mathfrak{s}(\mathbf{x}_m).$$

Proposition 3.9 from [GW] implies that for  $\Psi_1, \Psi_2 \in \pi_2(\mathbf{x}_1, \dots, \mathbf{x}_m)$ , we have  $\mathfrak{s}_H(\Psi_1) = \mathfrak{s}_H(\Psi_2)$  if and only if the difference  $\mathcal{D}(\Psi_1) - \mathcal{D}(\Psi_2)$  between the formal domains associated with  $\Psi_1$  and  $\Psi_2$  is a  $\mathbb{Z}$ -linear combination of doubly periodic domains, i.e. periodic domains for the Heegaard diagrams

$(\Sigma, \alpha^i, \alpha^{i+1}, \mathbf{z})$  for  $i = 1, \dots, m$ , where  $\alpha^{m+1} = \alpha^1$ .

Fix the intersection points

$$\mathbf{x}_i, \mathbf{x}'_i \in \mathbb{T}_i \cap \mathbb{T}_{i+1}, \quad i = 1, \dots, m-1 \quad \text{and} \quad \mathbf{x}_m, \mathbf{x}'_m \in \mathbb{T}_m \cap \mathbb{T}_1$$

We say that two homotopy class  $\Psi \in \pi_2(\mathbf{x}_1, \dots, \mathbf{x}_m)$  and  $\Psi' \in \pi_2(\mathbf{x}'_1, \dots, \mathbf{x}'_m)$  are *equivalent* if there exist Whitney disks  $\psi_i \in \pi_2(\mathbf{x}_i, \mathbf{x}'_i)$  for  $i = 1, \dots, m$  and  $\Psi$  is obtained from  $\Psi'$  by juxtaposition of the disk  $\psi_i$  at the vertex  $\mathbf{x}'_i$  for  $i = 1, \dots, m$ . Let  $\text{Polygons}_{\{1, \dots, m\}}$  denote the set of equivalence classes of such  $m$ -gons. We may thus re-state Proposition 3.9 of [GW] as follows.

**Proposition 2.6.** *The map from Proposition 2.5 refines to a one to one map*

$$\mathfrak{s}_H : \text{Polygons}_{\{1, \dots, m\}} \longrightarrow \text{Spin}^c(W_H).$$

Moreover, the image of  $\mathfrak{s}_H$  consists of the  $\text{Spin}^c$  structures whose restrictions to the boundary are realized by the intersection points.

For every index set

$$I = \{i_1 < \dots < i_p\} \subset \{1, \dots, m\}$$

with  $|I| \geq 3$  we may consider the cobordism  $W_I$  which corresponds to the compression of

$$W_H - \coprod_{i \notin I} U_i$$

along the edges  $\Sigma \times e_i \subset \Sigma \times \mathbb{D}_m$  with  $i \notin I$ . Thus  $W_I$  corresponds to the Heegaard diagram  $H_I = (\Sigma, \alpha^{i_1}, \dots, \alpha^{i_p}, \mathbf{u})$ . Correspondingly, we may define  $\text{Polygons}_I$  and there are restriction maps

$$\begin{array}{ccc} & \text{Spin}^c(W_H) & \\ & \downarrow & \\ \text{Polygons}_I & \xrightarrow{\mathfrak{s}_{H_I}} & \text{Spin}^c(W_I) \end{array}$$

These restriction maps associate to every  $\mathfrak{s} \in \text{Spin}^c(W_H)$  a coherent system of  $\text{Spin}^c$  structures for the Heegaard diagram  $H$  in the sense of Definition 6.2 from [AE].

Suppose that we have a pair of index sets  $I, J \subset \{1, \dots, m\}$  such that  $I = \{i_1 < i_2 < \dots < i_p\}$  and  $J = \{j_1 < j_2 < \dots < j_q\}$  for some  $1 \leq r \leq p-1$ . We then call the pair  $I, J$  *attachable*, and define

$$I \star J := \{i_1 < \dots < i_r = j_1 < j_2 < \dots < j_q = i_{r+1} < i_{r+2} < \dots < i_p\}.$$

We will denote  $r$  by  $r(I, J)$  for future reference. Suppose that  $I$  and  $J$  are attachable index sets as above, and we are given a  $p$ -gon  $\Psi_I$  and a  $q$ -gon  $\Psi_J$

$$\Psi_I \in \pi_2(\mathbf{x}_{i_1}, \dots, \mathbf{x}_{i_p}) \quad \text{and} \quad \Psi_J \in \pi_2(\mathbf{y}_{j_1}, \dots, \mathbf{y}_{j_q}),$$

where  $\mathbf{x}_{i_s} \in \mathbb{T}_{i_s} \cap \mathbb{T}_{i_{s+1}}$  and  $\mathbf{y}_{j_s} \in \mathbb{T}_{j_s} \cap \mathbb{T}_{j_{s+1}}$ . Furthermore, assume that  $\mathbf{x}_{i_r} = \mathbf{y}_{j_q}$ . Then we may *juxtapose*  $\Psi_I$  and  $\Psi_J$  to obtain the class of some  $(p+q-2)$ -gon, which will be denoted by  $\Psi_I \star \Psi_J$ .

Recall Definition 6.2 from [AE]:

**Definition 2.7.** A coherent system  $\mathfrak{t}$  of  $\text{Spin}^c$  structures for the Heegaard diagram  $H = (\Sigma, \boldsymbol{\alpha}^1, \dots, \boldsymbol{\alpha}^m, \mathbf{z})$  is a choice of classes

$$\mathcal{I} = \{[\mathfrak{s}_{H_I}(\Phi_I)] \in \text{Spin}^c(W_I) \mid I \subset \{1, \dots, m\}, |I| \geq 3\}$$

where the polygon  $\Phi_I$  corresponds to the index set  $I$  such that for attachable index sets  $I$  and  $J$  we have

$$[\Phi_{I \star J}] = [\Phi_I \star \Phi_J].$$

A coherent system of  $\text{Spin}^c$  structures for  $W_H$  is determined by the family of triangle classes

$$\left\{ \Phi_{1ij} \mid 1 < i < j \leq m \right\},$$

where  $\Phi_{1ij}$  corresponds to the index set  $I = \{1 < i < j\}$ . However, this family should satisfy the following compatibility property: for every triple  $1 < i < j < k \leq m$  of indices, there is a triangle class  $\Psi$  such that

$$(1) \quad \Psi_{1jk} \star \Psi_{1ij} = \Psi_{1ik} \star \Psi.$$

In particular, a coherent system of  $\text{Spin}^c$  structures over a Heegaard triple is determined by a  $\text{Spin}^c$  structure on the corresponding 4-manifold.

**2.4.  $\mathbb{A}$ -diagrams and holomorphic polygon maps.** Suppose that  $H = (\Sigma, \boldsymbol{\alpha}^1, \dots, \boldsymbol{\alpha}^m, \mathbf{z})$  is a balanced Heegaard diagram as before. Let  $\mathbb{A}$  be a commutative algebra over  $\mathbb{F}$  and  $\mathbf{u} : \mathbf{z} \rightarrow \mathbb{A}$  be a map. We will denote  $\mathbf{u}(z_i)$  by  $\mathbf{u}_i$ . Each set  $\boldsymbol{\alpha}^i$  of simple closed curves corresponds to a possibly disconnected surface  $\Sigma[i] = \coprod_{j=1}^{m^i} R_j^i$ , which is obtained from  $\Sigma$  by surgering out the simple closed curves in  $\boldsymbol{\alpha}^i$ . We denote the genus of  $R_j^i$  by  $g_j^i$ . Let us define

$$\mathbf{u}_j^i := \prod_{z_k \in R_j^i} \mathbf{u}_k \quad \text{and} \quad \mathbf{u}^i = \sum_{j=1}^{m^i} \mathbf{u}_j^i.$$

The map  $\mathbf{u} : \mathbf{z} \rightarrow \mathbb{A}$  is called a *representation* if the following are satisfied:

- (1) For any  $i, j \in \{1, \dots, m\}$  we have  $\mathbf{u}^i = \mathbf{u}^j$ .
- (2) If  $g_j^i > 0$  then  $\mathbf{u}_j^i = 0$ .

If  $\mathbf{u}$  is a representation, associated with every 2-chain  $\mathcal{D}$  on  $\Sigma$  with boundary on  $\boldsymbol{\alpha}^1 \cup \dots \cup \boldsymbol{\alpha}^m$  let  $n_i(\mathcal{D})$  denote the coefficient of  $\mathcal{D}$  at  $z_i$  and set

$$\mathbf{u}(\mathcal{D}) := \prod_{i=1}^{\kappa} \mathbf{u}_i^{n_i(\mathcal{D})} \in \mathbb{A}.$$



**Definition 2.8.** Let the  $\mathbb{A}$ -diagram  $H = (\Sigma, \alpha^1, \dots, \alpha^m, \mathbf{z})$ , the representation  $\mathbf{u} : \mathbf{z} \rightarrow \mathbb{A}$  and the coherent system  $\mathfrak{t}$  of  $\text{Spin}^c$  structures over  $H^\circ$  be as above. We call

$$(\Sigma, \alpha^1, \dots, \alpha^m, \mathbf{u} : \mathbf{z} \rightarrow \mathbb{A}, \mathfrak{t})$$

an  $\mathbb{A}$ -diagram if it is  $\mathfrak{t}$ -admissible, i.e. if the following admissibility condition is satisfied: For every index set  $I = \{i_1 < \dots < i_p\}$  and every doubly periodic domain

$$\mathcal{P} = \mathcal{P}_{i_1, i_2} + \mathcal{P}_{i_2, i_3} + \dots + \mathcal{P}_{i_{p-1}, i_p}$$

with  $\mathcal{P}_{i,j}$  a periodic domain on  $(\Sigma, \alpha^i, \alpha^j, \mathbf{z})$ , the following is true. If

$$\sum_{j=1}^p \left\langle c_1(\mathfrak{s}_{i_j, i_{j+1}}), H(\mathcal{P}_{i_j, i_{j+1}}) \right\rangle = 0$$

then either  $\mathbf{u}(\mathcal{P}) = 0$  in  $\mathbb{A}$  or the coefficient of the domain  $\mathcal{P}$  at some point  $w$  is negative. Here,  $\mathfrak{s}_{i,j}$  is the  $\text{Spin}^c$  structure on the 3-manifold  $M_{i,j}$  (corresponding to  $(\Sigma, \alpha^i, \alpha^j, \mathbf{z})$ ) which is determined by  $\mathfrak{t}$ .

Note that the admissibility condition only depends on the restrictions  $\mathfrak{s}_{i,j}$  of  $\mathfrak{t}$ . In particular, for every family  $\mathfrak{T}$  of  $\text{Spin}^c$  structures with the same restrictions  $\mathfrak{s}_{i,j}$ , we can speak of  $\mathbb{A}$ -diagrams

$$(\Sigma, \alpha^1, \dots, \alpha^m, \mathbf{u} : \mathbf{z} \rightarrow \mathbb{A}, \mathfrak{T}).$$

Let  $H = (\Sigma, \alpha^1, \dots, \alpha^m, \mathbf{u} : \mathbf{z} \rightarrow \mathbb{A}, \mathfrak{t})$  be an  $\mathbb{A}$ -diagram and choose an appropriate translation invariant family of almost complex structure  $J = \{J_z\}_{z \in \mathbb{D}_m}$ . For every pair of indices  $i < j$  we may thus define a chain complex

$$\text{CF}_J(\Sigma, \alpha^i, \alpha^j, \mathbf{u}, \mathfrak{s}_{i,j})$$

which is generated over  $\mathbb{A}$  by  $\mathbb{T}_i \cap \mathbb{T}_j$  and is equipped with the differential  $\partial_J$ . Associated with any subset

$$I = \{i_1 < \dots < i_p\} \subset \{1, \dots, m\}$$

of indices, we may then define a holomorphic polygon map

$$\begin{aligned} \mathfrak{f}_I : \bigotimes_{j=1}^{p-1} \text{CF}_J(\Sigma, \alpha^{i_j}, \alpha^{i_{j+1}}, \mathbf{u}, \mathfrak{s}_{i_j, i_{j+1}}) &\longrightarrow \text{CF}_J(\Sigma, \alpha^{i_1}, \alpha^{i_p}, \mathbf{u}, \mathfrak{s}_{i_1, i_p}). \\ \mathfrak{f}_I(\mathbf{x}_1 \otimes \mathbf{x}_2 \otimes \dots \otimes \mathbf{x}_{p-1}) &:= \sum_{\substack{\mathbf{x}_p \in \mathbb{T}_{i_1} \cap \mathbb{T}_{i_p} \\ \mathfrak{s}(\mathbf{x}_p) = \mathfrak{s}_{i_1, i_p}}} \sum_{\substack{\Psi \in \pi_2(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_p) \\ \mu(\Psi) = 3-p \\ \mathfrak{s}_{H_I}(\Psi) \in \mathfrak{t}}} (\mathfrak{m}(\Psi) \mathbf{u}(\Psi)) \cdot \mathbf{x}_p \end{aligned}$$

where  $\mu(\Psi)$  denotes the Maslov index of the polygon class  $\Psi$  and  $\mathfrak{m}(\Psi)$  is the count of points in  $\mathcal{M}(\Psi)$  modulo 2.

### 3. TANGLE COMPLEX AND THE ISSUE OF NATURALITY

In this section we address the issue of naturality in the sense of [JT] for the construction of [AE]. More precisely, we strengthen our result in [AE] and show that for any  $\mathbb{A}$ -module  $\mathbb{M}$ ,  $\mathrm{HF}^{\mathbb{M}}$  defines a functor from  $\mathbb{A}\text{-Tang}$  to  $\mathbb{A}\text{-Mod}$ . For the most part, the argument of [JT] for the naturality of sutured Floer homology may be copied here without significant modifications. More precisely, we need to go through the argument presented in Section 9 of [JT] and check that all statements remain valid, a task that is outlined in the present section. This would also help us fix our notation for the rest of the paper.

**3.1. The chain complex associated with a tangle.** Let us fix a commutative algebra  $\mathbb{A}$  over  $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$  and an  $\mathbb{A}$ -module  $\mathbb{M}$  as before. Consider an  $\mathbb{A}$ -tangle  $\mathcal{T} = [M, T, \mathfrak{s}, \mathbf{u}]$ . In this section, by an  $\mathbb{A}$ -diagram we mean a 5-tuple  $(\Sigma, \alpha, \beta, \mathbf{u} : \mathbf{z} \rightarrow \mathbb{A}, \mathfrak{s})$ , which is an  $\mathbb{A}$ -diagram in the sense of Definition 2.8. Each collection of disjoint simple closed curves on  $\Sigma$  (such as  $\alpha$  and  $\beta$ ) will be called *attaching sets*.

Each attaching set  $\gamma$  on the surface  $\Sigma$  determines a compression body  $C(\gamma)$  with  $\partial^- C(\gamma) = \Sigma$  and  $\partial^+ C(\gamma) = \Sigma[\gamma]$ , where  $\Sigma[\gamma]$  denotes the surface obtained by performing surgery on  $\Sigma$  along the curves in  $\gamma$ . Attaching sets  $\gamma$  and  $\gamma'$  on  $\Sigma$  are called *compression equivalent*, denoted by  $\gamma \sim \gamma'$ , if there is a diffeomorphism  $d : C(\gamma) \rightarrow C(\gamma')$  such that it is the identity on  $\Sigma = \partial^- C(\gamma) = \partial^- C(\gamma')$ . This gives an equivalence relation which descends to the isotopy classes, denoted by  $[\gamma]$ , of attaching sets  $\gamma$  on  $\Sigma$ . If  $\gamma \sim \gamma'$  then  $[\gamma]$  and  $[\gamma']$  are related by a sequence of handle slides, [JT, Lemma 2.11].

**Definition 3.1.** *We say that an  $\mathbb{A}$ -diagram  $(\Sigma, \alpha, \beta, \mathbf{u} : \mathbf{z} \rightarrow \mathbb{A}, \mathfrak{s})$  is a Heegaard diagram for an  $\mathbb{A}$ -tangle  $\mathcal{T}$  if  $\Sigma$  is a separating surface in  $M_{\mathcal{T}}$  which cuts  $T_{\mathcal{T}}$  transversely in  $\mathbf{z}$ ,  $\mathbf{u}$  is induced by  $\mathbf{u}_{\mathcal{T}}$  and  $\alpha$  and  $\beta$  are attaching sets of curves such that elements of  $\alpha$  and  $\beta$  bound disjoint disks on the two sides of  $\Sigma$ . Moreover, let  $\Sigma[\alpha]$  and  $\Sigma[\beta]$  be surfaces obtained from compressing  $\Sigma$  along the  $\alpha$  and  $\beta$  curves, respectively. Then,  $(\Sigma[\alpha], \mathbf{z})$  and  $(\Sigma[\beta], \mathbf{z})$  are isotopic relative to  $T_{\mathcal{T}}$  to*

$$(\partial^- M_{\mathcal{T}}, \partial^- T_{\mathcal{T}}) \quad \text{and} \quad (\partial^+ M_{\mathcal{T}}, \partial^+ T_{\mathcal{T}}),$$

*respectively. Finally, under the aforementioned isotopy, the  $\mathrm{Spin}^c$  class  $\mathfrak{s}$  corresponds to  $\mathfrak{s}_{\mathcal{T}}$ .*

Let  $(\Sigma, \alpha, \beta, \mathbf{u}, \mathfrak{s})$  be a Heegaard diagram for  $\mathcal{T}$ . If for the attaching sets  $\alpha'$  and  $\beta'$  we have  $[\alpha] = [\alpha']$  and  $[\beta] = [\beta']$ , and the  $\mathfrak{s}$ -admissibility condition is satisfied then  $(\Sigma, \alpha', \beta', \mathbf{u}, \mathfrak{s})$  is also a Heegaard diagram for  $\mathcal{T}$ . We may thus refer to the set of all such Heegaard diagrams as an *isotopy diagram*  $(\Sigma, A, B, \mathbf{u}, \mathfrak{s})$  for  $\mathcal{T}$ , where  $A = [\alpha]$  and  $B = [\beta]$ .

The isotopy diagrams

$$(\Sigma_1, A_1, B_1, \mathbf{u}_1 : \mathbf{z}_1 \rightarrow \mathbb{A}, \mathfrak{s}_1) \quad \text{and} \quad (\Sigma_2, A_2, B_2, \mathbf{u}_2 : \mathbf{z}_2 \rightarrow \mathbb{A}, \mathfrak{s}_2)$$

are called  $\alpha$ -equivalent if  $\Sigma_1 = \Sigma_2$ ,  $\mathbf{z}_1 = \mathbf{z}_2$ ,  $\mathbf{u}_1 = \mathbf{u}_2$  and  $B_1 = B_2$  while  $A_1 \sim A_2$  and under the natural correspondence between  $\text{Spin}^c$  structures,  $\mathfrak{s}_1 = \mathfrak{s}_2$ . Similarly, we may define  $\beta$ -equivalence. Stabilization and destabilization also induce operations on isotopy diagrams. Furthermore, any orientation preserving diffeomorphism  $d : \Sigma_1 \rightarrow \Sigma_2$  such that  $d(A_1) = A_2, d(B_1) = B_2, d(\mathbf{z}_1) = \mathbf{z}_2, d^*\mathfrak{s}_2 = \mathfrak{s}_1$  and  $\mathbf{u}_2 \circ d = \mathbf{u}_1$  induces a *diffeomorphism*

$$d : H_1 = (\Sigma_1, A_1, B_1, \mathbf{u}_1, \mathfrak{s}_1) \longrightarrow H_2 = (\Sigma_2, A_2, B_2, \mathbf{u}_2, \mathfrak{s}_2)$$

between isotopy diagrams.

Let  $\mathcal{G} = \mathcal{G}_{\mathcal{T}}$  denote the oriented *graph* (in the sense of Definition 2.21 from [JT]) whose vertices are isotopy diagrams for  $\mathcal{T}$  and for any two vertices  $H_1$  and  $H_2$  of  $\mathcal{G}$ , the set of edges connecting  $H_1$  to  $H_2$ , denoted by  $\mathcal{G}(H_1, H_2)$ , is a union of four sets

$$\mathcal{G}(H_1, H_2) = \mathcal{G}_{\alpha}(H_1, H_2) \sqcup \mathcal{G}_{\beta}(H_1, H_2) \sqcup \mathcal{G}_{\text{stab}}(H_1, H_2) \sqcup \mathcal{G}_{\text{diff}}(H_1, H_2),$$

defined as follows. The set  $\mathcal{G}_{\alpha}(H_1, H_2)$  (or  $\mathcal{G}_{\beta}(H_1, H_2)$ ) consists of a single arrow if  $H_1$  and  $H_2$  are  $\alpha$ -equivalent (or  $\beta$ -equivalent), otherwise it is empty. Similarly,  $\mathcal{G}_{\text{stab}}(H_1, H_2)$  consists of a single arrow if  $H_2$  is obtained from  $H_1$  by a stabilization or destabilization and  $\mathcal{G}_{\text{diff}}(H_1, H_2)$  contains an arrow corresponding to any diffeomorphism from  $H_1$  to  $H_2$ . Let  $\mathcal{G}_{\alpha}, \mathcal{G}_{\beta}, \mathcal{G}_{\text{stab}}$  and  $\mathcal{G}_{\text{diff}}$  denote the corresponding sub-graphs of  $\mathcal{G}$ . The sub-graphs  $\mathcal{G}_{\alpha}, \mathcal{G}_{\beta}$  and  $\mathcal{G}_{\text{diff}}$  are in fact categories when endowed with the obvious compositions. The graph  $\mathcal{G}$  is connected as an oriented graph.

Given an  $\mathbb{A}$ -diagram  $(\Sigma, \alpha, \beta, \mathbf{u}, \mathfrak{s})$ , every choice of the pair  $(j, J_s)$  of a complex structure  $j$  on  $\Sigma$  and a generic path  $J_s$  of perturbations of the induced complex structure on the symmetric product gives a chain complex

$$\text{CF}_{J_s}(\Sigma, \alpha, \beta, \mathbf{u}, \mathfrak{s}),$$

which is generated by the intersection points  $\mathbf{x} \in \mathbb{T}_{\alpha} \cap \mathbb{T}_{\beta}$  with  $\mathfrak{s}(\mathbf{x}) = \mathfrak{s}$ . We will usually drop the complex structure  $j$  on  $\Sigma$  from the notation. Given two different choices  $(j, J_s)$  and  $(j', J'_s)$  one obtains the chain map

$$\Phi_{J_s \rightarrow J'_s}^c : \text{CF}_{J_s}(\Sigma, \alpha, \beta, \mathbf{u}, \mathfrak{s}) \longrightarrow \text{CF}_{J'_s}(\Sigma, \alpha, \beta, \mathbf{u}, \mathfrak{s}).$$

The proof of Lemma 2.11 from [OS4] implies that the above chain map gives a corresponding isomorphism of  $\mathbb{A}$ -modules

$$\Phi_{J_s \rightarrow J'_s}^{\mathbb{M}} : \text{HF}_{J_s}^{\mathbb{M}}(\Sigma, \alpha, \beta, \mathbf{u}, \mathfrak{s}) \longrightarrow \text{HF}_{J'_s}^{\mathbb{M}}(\Sigma, \alpha, \beta, \mathbf{u}, \mathfrak{s}),$$

where  $\text{HF}_{J_s}^{\mathbb{M}}(\Sigma, \alpha, \beta, \mathbf{u}, \mathfrak{s})$  and  $\text{HF}_{J'_s}^{\mathbb{M}}(\Sigma, \alpha, \beta, \mathbf{u}, \mathfrak{s})$  are the homology groups of the chain complexes

$$\text{CF}_{J_s}(\Sigma, \alpha, \beta, \mathbf{u}, \mathfrak{s}) \otimes_{\mathbb{A}} \mathbb{M} \quad \text{and} \quad \text{CF}_{J'_s}(\Sigma, \alpha, \beta, \mathbf{u}, \mathfrak{s}) \otimes_{\mathbb{A}} \mathbb{M},$$

respectively. Moreover

$$\Phi_{J'_s \rightarrow J''_s} \circ \Phi_{J_s \rightarrow J'_s} = \Phi_{J_s \rightarrow J''_s}.$$

One may thus define

$$\mathrm{HF}^{\mathbb{M}}(\Sigma, \alpha, \beta, \mathbf{u}, \mathfrak{s}) = \frac{\coprod_{J_s} \mathrm{HF}_{J_s}^{\mathbb{M}}(\Sigma, \alpha, \beta, \mathbf{u}, \mathfrak{s})}{\sim}$$

where  $x \sim y$  if  $y = \Phi_{J_s \rightarrow J'_s}(x)$  for some  $J_s$  and  $J'_s$ . Since we will face several equivalence relations in the definition of  $\mathrm{HF}^{\mathbb{M}}(\mathcal{T})$  we will abuse the notation and denote all of them by  $\sim$ , leaving it to the reader to make the distinctions.

**3.2. Special Heegaard diagrams.** Let

$$H = (\Sigma, \alpha = \{\alpha_1, \dots, \alpha_\ell\}, \beta = \{\beta_1, \dots, \beta_\ell\}, \mathbf{u} : \mathbf{z} \rightarrow \mathbb{A}, \mathfrak{s}_0)$$

be an  $\mathbb{A}$ -diagram. Assume that  $[\alpha] \sim [\beta]$ . Then  $H$  determines an  $\mathbb{A}$ -tangle  $\mathcal{T}$  where  $(M, T) = (M_{\mathcal{T}}, T_{\mathcal{T}})$  is obtained from a product tangle by applying  $\ell$  zero surgeries. Let  $\mathfrak{s}_0 \in \mathrm{Spin}^c(M, T)$  be the torsion relative  $\mathrm{Spin}^c$  class on  $(M, T)$  i.e.  $c_1(\mathfrak{s}_0) = 0$  as an element of  $H^2(X, \mathbb{Z})$  where  $X = M - \mathrm{nd}(T)$ , and suppose that  $\mathfrak{s}_0 = [\mathfrak{s}_0]$ . In this case,  $H$  is called a *special Heegaard diagram*.

Let  $\mathbb{A}_T$  denote the algebra associated to the corresponding balanced sutured manifold for  $(M, T)$ , and  $\mathbf{u}_T : \mathbf{z} \rightarrow \mathbb{A}_T$  denote the corresponding representation. Let

$$\mathrm{HF}(M, T, \mathfrak{s}_0) = \mathrm{HF}(\Sigma, \alpha, \beta, \mathbf{u}_T, \mathfrak{s}_0)$$

denote the Heegaard Floer homology obtained using the coefficient ring  $\mathbb{A}_T$ . Note that for any  $\mathfrak{s} \in \mathrm{Spin}^c(M)$  with  $\mathfrak{s} \neq \mathfrak{s}_0$  we have  $\mathrm{HF}(M, T, \mathfrak{s}) = 0$ .

Suppose  $\mathbf{z} = \{z_1, \dots, z_\kappa\}$  and  $\mathbb{A}_T = \frac{\mathbb{Z}[\mathbf{u}_1, \dots, \mathbf{u}_\kappa]}{\mathcal{I}}$  where  $\mathbf{u}_i$  is the variable corresponding to  $z_i$  and  $\mathcal{I}$  is the ideal of relations. Consider  $a.\mathbf{x}, b.\mathbf{y} \in \mathrm{CF}(M, T, \mathfrak{s}_0)$  such that  $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$  and

$$a = \mathbf{u}_1^{a_1} \dots \mathbf{u}_\kappa^{a_\kappa} \quad \text{and} \quad b = \mathbf{u}_1^{b_1} \dots \mathbf{u}_\kappa^{b_\kappa}$$

are non-zero monomials in  $\mathbb{A}_T$ . Then we say a homotopy disk  $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$  connects  $a.\mathbf{x}$  to  $b.\mathbf{y}$  if  $a_i + n_{z_i}(\phi) = b_i$  for any  $i = 1, \dots, \kappa$ .

We define  $\mathrm{gr}(a.\mathbf{x}, b.\mathbf{y}) = \mu(\phi)$  for a homotopy disk  $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$  connecting  $a.\mathbf{x}$  to  $b.\mathbf{y}$ .

**Lemma 3.2.** *The map  $\mathrm{gr}$  is well-defined and induces a relative  $\mathbb{Z}$ -grading on*

$$\mathrm{CF}(M, T, \mathfrak{s}_0) = \mathrm{CF}(\Sigma, \alpha, \beta, \mathbf{z}, \mathfrak{s}_0).$$

*Furthermore, the subgroup of  $\mathrm{HF}(M, T, \mathfrak{s}_0)$  in top homological grading is isomorphic to  $\mathbb{A}_T$ .*

**Proof.** Let  $\phi_1, \phi_2 \in \pi_2(\mathbf{x}, \mathbf{y})$  be homotopy disks connecting  $a.\mathbf{x}$  to  $b.\mathbf{y}$ . Then,  $\mathcal{P} = \phi_1 - \phi_2$  would be a periodic domain such that  $n_{z_i}(\mathcal{P}) = 0$  for any  $i = 1, \dots, \kappa$ . Hence,

$$\mu(\phi_1) - \mu(\phi_2) = \mu(\mathcal{P}) = \langle c_1(\mathfrak{s}_0), H(\mathcal{P}) \rangle = 0.$$

In particular,  $\text{gr}$  is well-defined. It is not hard to show that  $\text{HF}(M, T, \mathfrak{s}_0)$  is invariant up to isomorphism in each relative grading. So the discussions in Section 6.2 of [AE] implies that the subgroup with the top grading is isomorphic to  $\mathbb{A}_T$ .  $\square$

Hence, the homology group in the top grading has a well-defined generator up to sign, called the *top generator*, in  $\text{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, u_T : \mathbf{z} \rightarrow \mathbb{A}, \mathfrak{s}_0)$ . The top generator is denoted by  $\Theta_{\alpha\beta}$ .

**3.3. Weak and strong Heegaard invariance.** Let us assume that

$$H = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\beta}', u, \mathfrak{t})$$

is an  $\mathbb{A}$ -diagram and  $\boldsymbol{\beta} \sim \boldsymbol{\beta}'$ . Furthermore, suppose that the restriction of  $\mathfrak{t}$  to  $M_{\beta, \beta'}$  is the distinguished  $\text{Spin}^c$  structure  $\mathfrak{s}_0$ , which was discussed in the previous subsection. With this restriction in place,  $\mathfrak{t}$  is determined by its restriction  $\mathfrak{s}$  to  $M_{\alpha, \beta}$ , and the induced  $\text{Spin}^c$  structure on  $M_{\alpha, \beta'}$  is in correspondence with  $\mathfrak{s}$  under the natural identification of  $M_{\alpha, \beta}$  with  $M_{\alpha, \beta'}$ . We will thus abuse the notation and denote the above  $\mathbb{A}$ -diagram by

$$(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\beta}', u, \mathfrak{s})$$

implying that the above restriction on  $\mathfrak{t}$  is imposed. Then, using the top generator  $\Theta_{\beta\beta'}$  we may define an isomorphism

$$\Phi_{\beta \rightarrow \beta'}^{\alpha} : \text{HF}^{\mathbb{M}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, u, \mathfrak{s}) \longrightarrow \text{HF}^{\mathbb{M}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}', u, \mathfrak{s}).$$

The arguments in the sequence of lemmas preceding Proposition 9.9 in [JT] may then be copied in our setup to show that if  $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, u, \mathfrak{s})$  and  $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}', u, \mathfrak{s})$  are Heegaard diagrams for  $\mathcal{T}$ , one may define a chain map and a well-defined induced isomorphism

$$\Phi_{\beta \rightarrow \beta'}^{\alpha} : \text{HF}^{\mathbb{M}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, u, \mathfrak{s}) \longrightarrow \text{HF}^{\mathbb{M}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}', u, \mathfrak{s}),$$

using the top generators and a detour to admissible triple diagrams (note that the above assumptions do not imply that the  $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\beta}', u, \mathfrak{s})$  is an  $\mathbb{A}$ -diagram). Furthermore, if  $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}'', u, \mathfrak{s})$  is a third  $\mathbb{A}$ -diagram, we get

$$(2) \quad \Phi_{\beta' \rightarrow \beta''}^{\alpha} \circ \Phi_{\beta \rightarrow \beta'}^{\alpha} = \Phi_{\beta \rightarrow \beta''}^{\alpha}.$$

Similarly, one may define  $\Phi_{\beta}^{\alpha \rightarrow \alpha'}$ . Thus, if  $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, u, \mathfrak{s})$ ,  $(\Sigma, \boldsymbol{\alpha}', \boldsymbol{\beta}, u, \mathfrak{s})$ ,  $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}', u, \mathfrak{s})$  and  $(\Sigma, \boldsymbol{\alpha}', \boldsymbol{\beta}', u, \mathfrak{s})$  are all  $\mathbb{A}$ -diagrams, one may define

$$\Phi_{\beta \rightarrow \beta'}^{\alpha \rightarrow \alpha'} = \Phi_{\beta \rightarrow \beta'}^{\alpha'} \circ \Phi_{\beta}^{\alpha \rightarrow \alpha'} = \Phi_{\beta'}^{\alpha \rightarrow \alpha'} \circ \Phi_{\beta \rightarrow \beta'}^{\alpha}.$$

Furthermore, we have

$$(3) \quad \begin{aligned} \Phi_{\beta' \rightarrow \beta''}^{\alpha' \rightarrow \alpha''} \circ \Phi_{\beta \rightarrow \beta'}^{\alpha \rightarrow \alpha'} &= \Phi_{\beta \rightarrow \beta''}^{\alpha \rightarrow \alpha''} \quad \text{and} \\ \Phi_{\beta \rightarrow \beta}^{\alpha \rightarrow \alpha} &= \Phi_{\beta \rightarrow \beta}^{\alpha} = \Phi_{\beta}^{\alpha \rightarrow \alpha} = \text{Id}_{\text{HF}^{\mathbb{M}}(\Sigma, \alpha, \beta, \mathbf{u}, \mathbf{s})}, \end{aligned}$$

see Lemma 9.11 from [JT].

If  $H = (\Sigma, A, B, \mathbf{u}, \mathbf{s})$  is an isotopy diagram, we denote by  $\mathcal{M}_H$  the set of all  $\mathbb{A}$ - diagrams corresponding to  $H$ , and we let

$$\text{HF}^{\mathbb{M}}(H) = \frac{\coprod_{(\Sigma, \alpha, \beta, \mathbf{u}, \mathbf{s}) \in \mathcal{M}_H} \text{HF}^{\mathbb{M}}(\Sigma, \alpha, \beta, \mathbf{u}, \mathbf{s})}{\sim}$$

where  $x \in \text{HF}^{\mathbb{M}}(\Sigma, \alpha, \beta, \mathbf{u}, \mathbf{s})$  and  $x' \in \text{HF}^{\mathbb{M}}(\Sigma, \alpha', \beta', \mathbf{u}, \mathbf{s})$  are equivalent if and only if  $x' = \Phi_{\beta \rightarrow \beta'}^{\alpha \rightarrow \alpha'}(x)$ . This construction associates a well-defined  $\mathbb{A}$ -module  $\text{HF}^{\mathbb{M}}(H)$  to every vertex  $H$  of  $\mathcal{G}_{\mathcal{T}}$ .

The next step in the argument of Juhász and Thurston is the construction of isomorphisms associated with the edges of  $\mathcal{G}_{\mathcal{T}}$ . If  $H = (\Sigma, A, B, \mathbf{u}, \mathbf{s})$  and  $H' = (\Sigma, A, B', \mathbf{u}, \mathbf{s})$  are  $\beta$ -equivalent, one may pick  $\mathbb{A}$ -diagram representatives  $(\Sigma, \alpha, \beta, \mathbf{u}, \mathbf{s})$  and  $(\Sigma, \alpha, \beta', \mathbf{u}, \mathbf{s})$  of  $H$  and  $H'$ , respectively. The formal proof of Lemma 9.17 from [JT] then implies that  $\Phi_{\beta \rightarrow \beta'}^{\alpha}$  descends to a well-defined isomorphism

$$\Phi_{B \rightarrow B'}^A : \text{HF}^{\mathbb{M}}(H) \longrightarrow \text{HF}^{\mathbb{M}}(H').$$

Similarly, the  $\alpha$ -equivalence from  $H = (\Sigma, A, B, \mathbf{u}, \mathbf{s})$  to  $H' = (\Sigma, A', B, \mathbf{u}, \mathbf{s})$  gives the well-defined isomorphism

$$\Phi_B^{A \rightarrow A'} : \text{HF}^{\mathbb{M}}(H) \longrightarrow \text{HF}^{\mathbb{M}}(H').$$

If  $d : H = (\Sigma, A, B, \mathbf{u}, \mathbf{s}) \rightarrow H' = (\Sigma', A', B', \mathbf{u}', \mathbf{s}')$  is a diffeomorphism, it gives a correspondence between complex structures, which in turn gives the isomorphism

$$d_* : \text{HF}^{\mathbb{M}}(H) \longrightarrow \text{HF}^{\mathbb{M}}(H'),$$

c.f. Definition 9.19 and Lemma 9.20 from [JT]. Finally, if  $(\Sigma', \alpha', \beta', \mathbf{u}', \mathbf{s}')$  is obtained from the  $\mathbb{A}$ -diagram  $(\Sigma, \alpha, \beta, \mathbf{u}, \mathbf{s})$  by stabilization, there is a correspondence between the homotopy classes of disks on the two sides. Furthermore, for suitable almost complex structures on the two diagrams, there is an isomorphism of chain complexes associated with the two diagrams. If  $H$  and  $H'$  denote the isotopy diagrams corresponding to the above two Heegaard diagrams for  $\mathcal{T}$  Lemma 9.21 from [JT] (which is basically Lemma 2.15 from [OS4]) may be used to construct the isomorphisms

$$\sigma_{H \rightarrow H'} : \text{HF}^{\mathbb{M}}(H) \longrightarrow \text{HF}^{\mathbb{M}}(H').$$

The construction of the isomorphisms associated with the edges of the graph  $\mathcal{G} = \mathcal{G}_{\mathcal{T}}$  implies that assigning  $\text{HF}^{\mathbb{M}}(H)$  to the vertices  $H$  of  $\mathcal{G}$  is a *weak Heegaard invariant*, and reproves the invariance of the quasi-isomorphism

type of the chain complex  $\mathrm{CF}^{\mathbb{M}}(\mathcal{T})$ .

Juhász and Thurston then show that the above assignment, which will be denoted by  $\mathrm{HF}^{\mathbb{M}}$ , is in fact a *strong Heegaard invariant* i.e. satisfies the following axioms:

- (1) **Functoriality:** The restriction of  $\mathrm{HF}^{\mathbb{M}}$  to the categories  $\mathcal{G}_\alpha, \mathcal{G}_\beta$  and  $\mathcal{G}_{\mathrm{diff}}$  is a functor to the category  $\mathbb{A}\text{-Mod}$  of  $\mathbb{A}$ -modules. Moreover, if  $\sigma : H \rightarrow H'$  is a stabilization and  $\sigma' : H' \rightarrow H$  is the corresponding destabilization then  $\mathrm{HF}^{\mathbb{M}}(\sigma') = \mathrm{HF}^{\mathbb{M}}(\sigma)^{-1}$ .
- (2) **Commutativity:** For every distinguished rectangle

$$\begin{array}{ccc} H_1 & \xrightarrow{\mathfrak{h}_1} & H_2 \\ \mathfrak{h}_2 \downarrow & & \downarrow \mathfrak{h}_3 \\ H_3 & \xrightarrow{\mathfrak{h}_4} & H_4 \end{array}$$

in  $\mathcal{G}$  we have  $\mathrm{HF}^{\mathbb{M}}(\mathfrak{h}_3) \circ \mathrm{HF}^{\mathbb{M}}(\mathfrak{h}_1) = \mathrm{HF}^{\mathbb{M}}(\mathfrak{h}_4) \circ \mathrm{HF}^{\mathbb{M}}(\mathfrak{h}_2)$ . Here, distinguished rectangles are the rectangles of the above type such that one of the following is the case:

- The horizontal arrows are  $\alpha$ -equivalences while the verticals are  $\beta$ -equivalences.
  - Both horizontal arrows are  $\alpha$ - or  $\beta$ -equivalences, while the vertical arrows are stabilizations.
  - Both horizontal arrows are  $\alpha$ - or  $\beta$ -equivalences, while the vertical arrows are diffeomorphisms with the same restriction on the surface.
  - The square corresponds to the two possible ways of performing disjoint stabilizations.
  - The horizontal arrows are diffeomorphisms and the vertical arrows are stabilizations which correspond to one another via the diffeomorphisms.
- (3) **Continuity:** If  $H$  is a vertex of  $\mathcal{G}$  and  $d \in \mathcal{G}_{\mathrm{diff}}(H, H)$  is a diffeomorphism isotopic to the identity, then  $\mathrm{HF}^{\mathbb{M}}(d) = \mathrm{Id}_{\mathrm{HF}^{\mathbb{M}}(H)}$ .
  - (4) **Handleswap invariance:** For every simple handle swap

$$\begin{array}{ccc} & H_1 & \\ \mathfrak{h}_3 \uparrow & \searrow \mathfrak{h}_1 & \\ H_3 & \xleftarrow{\mathfrak{h}_2} & H_2 \end{array}$$

in the sense of [JT, Definition 2.32] in  $\mathcal{G}$  we have

$$\mathrm{HF}^{\mathbb{M}}(\mathfrak{h}_3) \circ \mathrm{HF}^{\mathbb{M}}(\mathfrak{h}_2) \circ \mathrm{HF}^{\mathbb{M}}(\mathfrak{h}_1) = \mathrm{Id}_{\mathrm{HF}^{\mathbb{M}}(H_1)}.$$

Functoriality follows from equations (2) and (3) for  $\alpha$ - and  $\beta$ -equivalences, and is a consequence of the definition for the diffeomorphisms. Moreover, the equality  $\text{HF}_{\mathbb{M}}(\sigma') = \text{HF}_{\mathbb{M}}(\sigma)^{-1}$  is a consequence of the definition.

Our earlier considerations proves the axiom of commutativity for the first three types of distinguished rectangles. The commutativity of a diagram of the fourth type is satisfied in the level of chain complexes if the  $\mathbb{A}$ -diagrams (representing  $H_i$ ), and the complex structures on the surfaces are chosen correctly. Similarly and following the argument of Subsection 9.2 from [JT], for suitable  $\mathbb{A}$ -diagrams representing the isotopy diagrams  $H_i$  and the correct choice of almost complex structures, the commutativity of the diagrams of the fifth type is satisfied in the level of chain complexes.

One may then copy the proof of Proposition 9.23 from [JT] and show that if for the  $\mathbb{A}$ -diagram  $(\Sigma, \alpha, \beta, \mathbf{u}, \mathfrak{s})$  the map  $d : \Sigma \rightarrow \Sigma$  is isotopic to identity, then  $d_* = \Phi_{\beta \rightarrow \beta'}^{\alpha \rightarrow \alpha'}$ , where  $\alpha' = d(\alpha)$  and  $\beta' = d(\beta)$ . This implies that the induced isomorphism

$$d_* : \text{HF}^{\mathbb{M}}(\Sigma, A, B, \mathbf{u}, \mathfrak{s}) \longrightarrow \text{HF}^{\mathbb{M}}(\Sigma, A, B, \mathbf{u}, \mathfrak{s})$$

is the identity, by our definition of  $\text{HF}_{\mathbb{M}}(\Sigma, A, B, \mathbf{u}, \mathfrak{s})$ . This completes the proof of continuity.

Finally, let  $H_1, H_2$  and  $H_3$  denote the isotopy diagrams corresponding to a handleswap. Choose  $\mathbb{A}$ -diagrams  $(\Sigma, \alpha_i, \beta_i, \mathbf{u})$  representing  $H_i$ ,  $i = 1, 2, 3$ . The argument given for handleswap invariance in Subsection 9.3 of [JT] regards the region corresponding to the handleswap in the aforementioned Heegaard diagrams as a genus 2 subsurface  $\Sigma^0$  of the connected sum of  $\Sigma^0$  with another surface  $\Sigma^1$ , such that the Heegaard diagrams are identical on  $\Sigma^1$ , all markings in  $\mathbf{z}$  lie on  $\Sigma^1$ , and two curves from each one of the attaching sets  $\alpha_i$  and  $\beta_i$  are on  $\Sigma^0$ . Lemmas 9.25 and 9.28 from [JT] (which stay correct in our more general framework) imply that every triangle class which contributes to  $\text{HF}^{\mathbb{M}}(\mathfrak{h}_1)$  or  $\text{HF}^{\mathbb{M}}(\mathfrak{h}_2)$  may be decomposed as a disjoint union of a triangle class on  $\Sigma^1$  which does not pass through the connected sum region and a small triangle class on  $\Sigma^0$ . The proof of Proposition 9.24 from [JT] thus goes through without difficulty, completing the proof of handleswap invariance.

Let  $\mathfrak{h}$  denote an oriented path in  $\mathcal{G}_{\mathcal{T}}$  from the isotopy diagram  $H$  to the isotopy diagram  $H'$ . Composing the isomorphisms corresponding to the edges in  $\mathfrak{h}$  we obtain an isomorphism

$$\text{HF}^{\mathbb{M}}(\mathfrak{h}) : \text{HF}^{\mathbb{M}}(H) \longrightarrow \text{HF}^{\mathbb{M}}(H').$$

If  $\mathfrak{h}$  and  $\mathfrak{h}'$  are different oriented paths from  $H$  to  $H'$ , Theorem 2.39 from [JT], which is probably the most important contribution of the aforementioned paper, implies that  $\text{HF}^{\mathbb{M}}(\mathfrak{h}) = \text{HF}^{\mathbb{M}}(\mathfrak{h}')$ , since  $\text{HF}^{\mathbb{M}}$  is a strong Heegaard



invariant. In other words, associated with the vertices  $H$  and  $H'$  of  $\mathcal{G}_{\mathcal{T}}$  there is a well-defined isomorphism

$$\mathrm{HF}_{H \rightarrow H'}^{\mathbb{M}} : \mathrm{HF}^{\mathbb{M}}(H) \longrightarrow \mathrm{HF}^{\mathbb{M}}(H').$$

It is clear that

$$\mathrm{HF}_{H' \rightarrow H''}^{\mathbb{M}} \circ \mathrm{HF}_{H \rightarrow H'}^{\mathbb{M}} = \mathrm{HF}_{H \rightarrow H''}^{\mathbb{M}}.$$

One may thus define

$$\mathrm{HF}^{\mathbb{M}}(\mathcal{T}) := \frac{\coprod_{H \in |\mathcal{G}_{\mathcal{T}}|} \mathrm{HF}^{\mathbb{M}}(H)}{\sim},$$

where  $x$  is equivalent to  $y$  if  $y = \mathrm{HF}_{H \rightarrow H'}^{\mathbb{M}}(x)$ . Thus, associated with every  $\mathbb{A}$ -tangle  $\mathcal{T}$  and  $\mathbb{A}$ -module  $\mathbb{M}$  we obtain a well-defined  $\mathbb{A}$ -module  $\mathrm{HF}^{\mathbb{M}}(\mathcal{T})$ .

If  $d : \mathcal{T} = [M, T, \mathbf{u}, \mathfrak{s}] \rightarrow [M', T', \mathbf{u}', \mathfrak{s}']$  is a diffeomorphism of  $\mathbb{A}$ -tangles, pick an isotopy diagram  $H = (\Sigma, A, B, \mathbf{u}, \mathfrak{s})$  for  $\mathcal{T}$ , where  $\Sigma$  is a surface in  $M$ . The map  $d$  takes  $\Sigma$  to a surface  $\Sigma'$  in  $M'$  and the markings  $\mathbf{z}$  to a set  $\mathbf{z}'$  of markings such that  $\mathbf{z}' = \Sigma' \cap T'$ . Furthermore, we obtain a  $\mathrm{Spin}^c$  structure  $\mathfrak{s}'$  for  $(\Sigma', \alpha', \beta', \mathbf{z}')$  which corresponds to  $\mathfrak{s}$ . We may define  $\mathbf{u}' : \mathbf{z}' \rightarrow \mathbb{A}$  as  $\mathbf{u} \circ d^{-1}$ . We thus obtain the isotopy diagram  $H' = d(H)$  for  $\mathcal{T}'$ . The isomorphism

$$\mathrm{HF}^{\mathbb{M}}(d) : \mathrm{HF}^{\mathbb{M}}(H) \longrightarrow \mathrm{HF}^{\mathbb{M}}(H')$$

associated with the diffeomorphism  $d$  from  $H$  to  $H'$ , induces a well-defined isomorphism

$$\mathrm{HF}^{\mathbb{M}}(d) : \mathrm{HF}^{\mathbb{M}}(\mathcal{T}) \longrightarrow \mathrm{HF}^{\mathbb{M}}(\mathcal{T}').$$

The above considerations imply the following theorem.

**Theorem 3.3.** *For every algebra  $\mathbb{A}$  over  $\mathbb{F}$  and every  $\mathbb{A}$ -module  $\mathbb{M}$ , assigning  $\mathrm{HF}^{\mathbb{M}}(\mathcal{T})$  to the  $\mathbb{A}$ -tangle  $\mathcal{T}$  in  $\mathbb{A}\text{-Tang}$  gives a functor*

$$\mathrm{HF}^{\mathbb{M}} : \mathbb{A}\text{-Tang} \longrightarrow \mathbb{A}\text{-Mod}.$$

**3.4. The action of  $\Lambda^*(H_1(M, \mathbb{Z})/\mathrm{Tors})$ .** As in the usual setup of the Heegaard Floer homology, there is a natural action of  $\Lambda^*(H_1(M_{\mathcal{T}}, \mathbb{Z})/\mathrm{Tors})$  on  $\mathrm{HF}^{\mathbb{M}}(\mathcal{T})$  as follows. Let us assume that  $H = (\Sigma, \alpha, \beta, \mathbf{u} : \mathbf{z} \rightarrow \mathbb{A}, \mathfrak{s})$  is an  $\mathbb{A}$ -diagram for the  $\mathbb{A}$ -tangle  $\mathcal{T} = [M, T, \mathbf{u}, \mathfrak{s}]$ . First of all, as discussed in Subsection 2.4 of [OS2], there is a homotopy long exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \pi_1(\Omega(\mathbb{T}_{\alpha}, \mathbb{T}_{\beta})) \longrightarrow \pi_1(\mathbb{T}_{\alpha} \times \mathbb{T}_{\beta}) \longrightarrow \pi_1(\mathrm{Sym}^g(\Sigma)).$$

Here  $\Omega(\mathbb{T}_{\alpha}, \mathbb{T}_{\beta})$  denotes the space of paths in  $\mathrm{Sym}^g(\Sigma)$  joining  $\mathbb{T}_{\alpha}$  to  $\mathbb{T}_{\beta}$ . Under the identification of  $\pi_1(\mathrm{Sym}^g(\Sigma))$  with  $H^1(\Sigma, \mathbb{Z})$ ,  $\pi_1(\mathbb{T}_{\alpha})$  and  $\pi_1(\mathbb{T}_{\beta})$  correspond to  $H^1(C(\alpha), \mathbb{Z})$  and  $H^1(C(\beta), \mathbb{Z})$ , respectively. After comparing the above exact sequence with the cohomology long exact sequence for the decomposition  $M = C(\alpha) \cup_{\Sigma} C(\beta)$ , we obtain the short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \pi_1(\Omega(\mathbb{T}_{\alpha}, \mathbb{T}_{\beta})) \longrightarrow H^1(M, \mathbb{Z}) \longrightarrow 0.$$

Applying  $\text{Hom}(-, \mathbb{Z})$  to the above short exact sequence we obtain

$$0 \longrightarrow H_1(M, \mathbb{Z})/\text{Tors} \longrightarrow H^1(\Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta), \mathbb{Z}) \longrightarrow \mathbb{Z}.$$

Every element  $\zeta \in H_1(M, \mathbb{Z})/\text{Tors}$  may thus be realized as an element of  $H^1(\Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta), \mathbb{Z})$ , which is realized by a 1-cocycle  $Z(\zeta) \in Z^1(\Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta), \mathbb{Z})$  in the space of paths connecting  $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ . If  $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$  is the homotopy class of a Whitney disk, it may be viewed as an arc in  $\Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta)$  which connects the constant path at  $\mathbf{x}$  to the constant path at  $\mathbf{y}$ . The evaluation of  $Z(\zeta)$  over this path gives a value  $n_{Z(\zeta)}(\phi)$ . Correspondingly, we may define a map

$$\begin{aligned} A_{Z(\zeta)} : \text{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{u}, \mathbf{s}) &\rightarrow \text{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{u}, \mathbf{s}) \\ A_{Z(\zeta)}(\mathbf{x}) &:= \sum_{\substack{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta \\ \mathbf{s}(\mathbf{y}) = \mathbf{s}}} \sum_{\phi \in \pi_2^1(\mathbf{x}, \mathbf{y})} n_{Z(\zeta)}(\phi) \mathbf{m}(\phi) \mathbf{u}(\phi) \mathbf{y}. \end{aligned}$$

The proof of Lemma 4.18 from [OS2] implies that  $A_{Z(\zeta)}$  is a chain map and the proof of Lemma 4.19 from [OS2] implies that if  $Z(\zeta)$  is a coboundary then  $A_{Z(\zeta)}$  is chain homotopic to zero. The map induced by  $A_{Z(\zeta)}$  on homology is thus independent of the choice of the cocycle  $Z(\zeta)$ , and may thus be represented by

$$A_\zeta : \text{HF}^{\mathbb{M}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{u}, \mathbf{s}) \rightarrow \text{HF}^{\mathbb{M}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{u}, \mathbf{s}).$$

The proof of Proposition 4.17 from [OS2] may then be copied to show that  $A_\zeta \circ A_\zeta = 0$ . Consequently, we obtain an action of the exterior algebra  $\Lambda^*(H_1(M, \mathbb{Z})/\text{Tors})$  on  $\text{HF}^{\mathbb{M}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{u}, \mathbf{s})$ .

We may then follow the steps toward weak and strong Heegaard invariance of the functor  $\text{HF}^{\mathbb{M}}$ , and observe that all the isomorphisms which correspond to the edges of the graph  $\mathcal{G}_{\mathcal{T}}$  preserve the action of  $\Lambda^*(H_1(M, \mathbb{Z})/\text{Tors})$  constructed above. This observation implies the following proposition.

**Proposition 3.4.** *For every  $\mathbb{A}$ -tangle  $\mathcal{T}$  and every  $\mathbb{A}$ -module  $\mathbb{M}$ , there is a natural action of  $\Lambda^*(H_1(M_{\mathcal{T}}, \mathbb{Z})/\text{Tors})$  on  $\text{HF}^{\mathbb{M}}(\mathcal{T})$ .*

## 4. PARAMETRIZED CERF DECOMPOSITION

**4.1. Parametrized elementary cobordism.** Let  $(M, T)$  be a balanced tangle. For  $k = 0, 1, 2$ , a framed  $k$ -sphere  $\mathbb{S}$  in  $(M, T)$  is an embedding of  $S^k \times D^{3-k}$  in  $M - T$ . In other word, it is an embedded  $k$ -sphere  $S = S^k \times \{0\}$  in  $M - T$  together with a trivialization  $\nu$  of its normal bundle. We will denote by

$$\mathcal{W}(\mathbb{S}) = (W(\mathbb{S}), F(\mathbb{S}))$$

the cobordism obtained by attaching a  $k$ -handle to  $M \times [0, 1]$  along  $\mathbb{S} \times \{1\}$  to construct  $W(\mathbb{S})$  and setting  $F(\mathbb{S}) = T \times [0, 1] \subset W(\mathbb{S})$ . Thus  $\mathcal{W}(\mathbb{S})$  is a cobordism from  $(M, T)$  to  $(M(\mathbb{S}), T)$ , where  $M(\mathbb{S})$  is obtained by surgery on  $M$  along  $\mathbb{S}$ .

Similarly, a framed arc  $\mathbb{I}$  in  $(M, T)$  is an embedding of  $D^2 \times D^1$  in  $M$  such that

$$\mathbb{I}^{-1}(T) = (\{x = 0\} \times \{-1\}) \amalg (\{x = 0\} \times \{1\}),$$

where  $(x, y)$  denotes the standard coordinate system on  $D^2$  (as a subset of  $\mathbb{R}^2$ ). Moreover,  $\mathbb{I}$  is called *orientation preserving* if the restriction of  $\mathbb{I}$  to

$$\{x = 0\} \times \{-1\} \amalg \{x = 0\} \times \{1\} \subset \partial(\{x = 0\} \times [-1, 1]),$$

which is equipped with the boundary orientation, is orientation preserving as a map to  $T$ . We think of  $\mathbb{I}$  as an embedded arc  $\sigma = \{0\} \times D^1$  connecting two points  $\partial\sigma \cap T$ , together with a trivialization  $\nu$  of its normal bundle in  $M$  which is at the end points compatible with the trivialization induced from the orientation of  $T$ . Associated with a framed arc  $\mathbb{I}$ , let  $T(\mathbb{I}) \subset M$  be the properly embedded 1-manifold obtained by doing band surgery on  $T$  along  $\mathbb{I}$  i.e.

$$T(\mathbb{I}) = (T - \mathbb{I} \cap T) \bigcup \mathbb{I}((\{x = -1\} \cup \{x = 1\}) \times [-1, 1]).$$

Note that if  $\mathbb{I}$  is orientation preserving, then the orientation on  $T$  induces an orientation on  $T(\mathbb{I})$ . Moreover, for every set  $\mathbb{I} = \{\mathbb{I}_1, \dots, \mathbb{I}_n\}$  of framed arcs in  $(M, T)$ , denote the properly embedded 1-manifold constructed by doing surgery on  $T$  along the framed arcs  $\mathbb{I}_1, \dots, \mathbb{I}_n$  by  $T(\mathbb{I})$ . Thus, if  $\mathbb{I}$  is a single framed arc,  $T(\mathbb{I})$  and  $T(\{\mathbb{I}\})$  are the same object.

**Definition 4.1.** A set  $\mathbb{I} = \{\mathbb{I}_1, \dots, \mathbb{I}_n\}$  of framed arcs in  $(M, T)$  is called acceptable if for any  $i$ ,  $\mathbb{I}_i$  is orientation preserving and  $(M, T(\mathbb{I}))$  is a tangle.

Let  $\mathbb{I} = \{\mathbb{I}_1, \dots, \mathbb{I}_n\}$  be an acceptable set of framed arcs in  $(M, T)$ . Corresponding to  $\mathbb{I}$ , we construct a cobordism  $\mathcal{W}(\mathbb{I}) = (W(\mathbb{I}), F(\mathbb{I}))$  from  $(M, T)$  to  $(M, T(\mathbb{I}))$  by attaching a *standard pair* to  $(M \times [0, 1], T \times [0, 1])$  along  $\mathbb{I} \subset M \times \{1\}$  as follows. A standard pair is a pair  $\mathcal{H} = (H, B)$ , which is

identified in  $\mathbb{R}^4$  via

$$H := D^2 \times D^1 \times D^1 = \left\{ (x, y, z, t) \in \mathbb{R}^4 \mid \begin{array}{l} x^2 + y^2 \leq 1 \\ z, t \in [-1, 1] \end{array} \right\} \quad \text{and}$$

$$B = \left\{ (x, y, z, t) \in D^2 \times D^1 \times D^1 \mid \begin{array}{l} (t+1)y^2 + (t-1)z^2 = 2t \\ x = 0 \end{array} \right\}$$

See Figure 1 for a picture of the projection of a standard pair over the 3-dimensional box  $\{x = 0\} \times D^1 \times D^1$ .

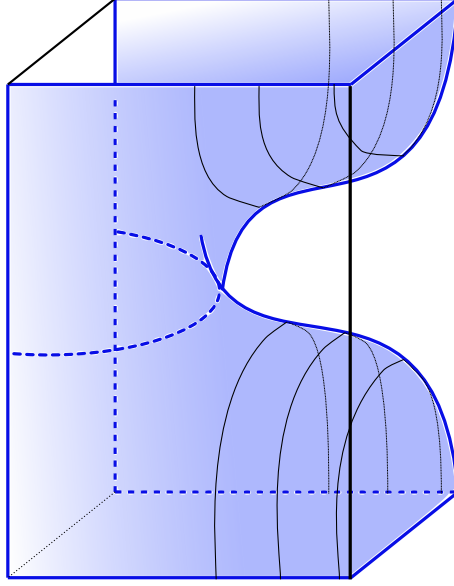


FIGURE 1. The standard pair, which is pictured in the cube  $\{x = 0\} \times [-1, 1]^3$ .

Let  $\mathcal{W}(\mathbb{I}) = (W(\mathbb{I}), F(\mathbb{I}))$  denote the cobordism

$$\frac{(M \times [0, 1], T \times [0, 1]) \amalg (\amalg_{i=1}^n \mathcal{H}_i)}{\left\{ \mathbb{I}_i (D^2 \times D^1) \times \{1\} \sim (D^2 \times D^1 \times \{-1\}) \subset H_i \mid \text{for any } 1 \leq i \leq n \right\}}$$

after smoothing the corners along

$$\begin{aligned} \partial(\mathbb{I}_i(D^2 \times D^1) \times \{1\}) &\sim \partial(D^2 \times D^1 \times \{-1\}) \subset H_i \quad \text{and} \\ \partial(D^2 \times D^1 \times \{1\}) &\subset H_i. \end{aligned}$$

Here  $\mathcal{H}_i = (H_i, B_i)$  are copies of the standard pair  $(H, B)$  for  $i = 1, \dots, n$ . Note that  $W(\mathbb{I}) \simeq M \times [0, 1]$ .

**Definition 4.2.** A stable cobordism  $\mathcal{W} = (W, F)$  from the tangle  $(M, T)$  to the tangle  $(M', T')$  is called elementary, if for a framed sphere  $\mathbb{S}$  or an acceptable set of framed arcs  $\mathbb{I}$  in  $(M, T)$ ,  $\mathcal{W}$  is diffeomorphic to the corresponding cobordism  $\mathcal{W}(\mathbb{S})$  or  $\mathcal{W}(\mathbb{I})$ .

Moreover, a parametrized elementary cobordism is an elementary cobordism  $\mathcal{W}$  as above, accompanied with one of the following:

- (1) A framed sphere  $\mathbb{S} \subset M$ , together with the isotopy class of a diffeomorphism

$$d : (M(\mathbb{S}), T) \rightarrow (M', T')$$

so that for a diffeomorphism  $D : \mathcal{W}(\mathbb{S}) \rightarrow \mathcal{W}$  with  $D|_{(M, T)} = \text{Id}$ , we have  $d = D|_{(M(\mathbb{S}), T)}$ .

- (2) A framed arc  $\mathbb{I} \subset M$ , together with the isotopy class of a diffeomorphism

$$d : (M, T(\mathbb{I})) \rightarrow (M', T')$$

so that for a diffeomorphism  $D : \mathcal{W}(\mathbb{I}) \rightarrow \mathcal{W}$  with  $D|_{(M, T)} = \text{Id}$ , we have  $d = D|_{(M, T(\mathbb{I}))}$ .

Note that in the above definition we might also have  $\mathbb{S} = \emptyset$ .

**Definition 4.3.** A parametrized Cerf decomposition of a stable cobordism  $\mathcal{W} = (W, F)$  from a tangle  $(M, T)$  to a tangle  $(M', T')$  is a decomposition

$$\mathcal{W} = \mathcal{W}_1 \cup_{(M_1, T_1)} \dots \cup_{(M_{m-1}, T_{m-1})} \mathcal{W}_m,$$

where  $\mathcal{W}_i = (W_i, F_i)$  is a parametrized elementary cobordism from the tangle  $(M_{i-1}, T_{i-1})$  to the tangle  $(M_i, T_i)$ ,  $(M_0, T_0) = (M, T)$  and  $(M_m, T_m) = (M', T')$ . For every  $i = 1, \dots, m$ , depending on the type of  $\mathcal{W}_i$ , we let  $(\mathbb{S}_i, d_i)$  or  $(\mathbb{I}_i, d_i)$  denote its parametrization.

**4.2. Parametrized Morse data.** Let  $\mathcal{W} = (W, F)$  be a cobordism from  $(M, T)$  to  $(M', T')$ . A function  $G : W \rightarrow [a, b]$  is called a Morse function on  $\mathcal{W}$ , if  $G$  has no critical points in a neighborhood of  $\partial W \cup F$ , both  $G$  and  $g = G|_F$  are Morse and

$$G^{-1}(a) = M, \quad G^{-1}(b) = M' \quad \text{and} \quad G|_{\partial_h W} = \pi_2,$$

where  $\pi_2$  is the projection over the second factor under an identification

$$(\partial_h W, \partial_h F) = (\partial M, \partial T) \times [a, b].$$

The set of critical points of  $G$  on  $\mathcal{W}$ ,  $\text{Crit}_{\mathcal{W}}(G)$ , is defined as

$$\text{Crit}_{\mathcal{W}}(G) := \text{Crit}_W(G) \cup \text{Crit}(g)$$

where  $\text{Crit}_W(G)$  and  $\text{Crit}(g)$  are the sets of critical points of  $G$  on  $W$  and  $g = G|_F$  on  $F$ , respectively. A Morse function  $G$  on  $\mathcal{W}$  is called *proper* if it has distinct values at its critical points on  $\mathcal{W}$ .  $G$  is called *indefinite* if both  $G$  and  $g$ , as Morse functions on  $W$  and  $F$  respectively, are indefinite i.e. have no critical points with the minimal and maximal index. Therefore, if  $G$  is indefinite on  $\mathcal{W}$ ,  $g$  has only critical points of index 1, while the critical points of  $G$  are of indices 1, 2 or 3.

**Definition 4.4.** Let  $G : W \rightarrow [a, b]$  be a Morse function on the cobordism  $\mathcal{W} = (W, F)$ . A vector field  $\xi$  on  $W$  is called an embedded gradient-like vector field for  $G$  if it satisfies the following conditions:

- (1) For every  $p \in W$  which is not in  $\text{Crit}_W(G)$ ,  $dG_p(\xi_p) > 0$ .
- (2) The vector field  $\xi$  is tangent to both  $F$  and  $\partial_h W$ .
- (3) For any critical point  $p \in \text{Crit}_W(G)$  there is an open neighborhood  $U \subset W$  of  $p$  together with a positively oriented local coordinate system  $(x_1, x_2, x_3, x_4)$  centered at  $p$  such that

$$G(x_1, x_2, x_3, x_4) = G(p) \pm x_1^2 \pm x_2^2 \pm x_3^2 \pm x_4^2$$

$$\xi(x_1, x_2, x_3, x_4) = (\pm x_1, \pm x_2, \pm x_3, \pm x_4)$$

- (4) There is an open neighborhood  $U \subset W$  of every critical point  $p$  of  $g$  and a positively oriented local coordinate system  $(x_1, x_2, y_1, y_2)$  centered at  $p$ , such that

$$U \cap F = \{(x_1, x_2, y_1, y_2) \in U \mid y_1 = y_2 = 0\},$$

$$G(x_1, x_2, y_1, y_2) = G(p) \pm x_1^2 \pm x_2^2 + y_1 \quad \text{and}$$

$$\xi = (\pm x_1, \pm x_2, y_1^2 + y_2^2, 0).$$

**Definition 4.5.** A Morse datum for  $\mathcal{W}$  is a triple  $\mathfrak{M} = (G, \underline{b}, \xi)$ , where  $\underline{b} = (a = b_0 < \dots < b_m = b) \in \mathbb{R}^{m+1}$ , is an ordered  $(m+1)$ -tuple of regular values for the proper Morse function  $G : W \rightarrow [a, b]$  on  $\mathcal{W}$ , and  $\xi$  is an embedded gradient like vector field for  $G$ . Over each interval  $(b_i, b_{i+1})$ ,  $G$  has at most one critical point in  $W$ , and if it has a critical point in  $G^{-1}(b_i, b_{i+1})$  then  $g$  has no critical point over this interval.

The Morse datum  $(G, \underline{b}, \xi)$  is called good if  $g$  is indefinite and  $g^{-1}(b_i)$  is a union of arcs connecting  $\partial_h^- W$  to  $\partial_h^+ W$  for  $i = 0, \dots, m$ .

Any good Morse datum  $\mathfrak{M} = (G, \underline{b}, \xi)$  induces a parametrized Cerf decomposition  $\mathfrak{C}(G, \underline{b}, \xi)$  of  $\mathcal{W}$  by taking  $W_i = G^{-1}[b_{i-1}, b_i]$ ,  $F_i = g^{-1}[b_{i-1}, b_i]$  and  $(M_i, T_i) = (G^{-1}(b_i), g^{-1}(b_i))$ . If  $G$  has a critical point  $p \in G^{-1}[b_{i-1}, b_i]$  of index  $k_i$ , the descending flow of  $\xi$  maps  $p$  to a  $k_i - 1$  dimensional embedded sphere  $S_i \subset G^{-1}(b_{i-1})$ , which is disjoint from  $T_{i-1}$ . Moreover, using a positive local coordinate system in the neighborhood of  $p$  such that  $G$  and  $\xi$  have the standard structure, we obtain a framing for  $S_i$ . As in Remark 2.12 of [Juh1], if  $k_i = 0, 4$  the framed sphere  $\mathbb{S}_i$  does not depend on the choice of the local coordinates and is uniquely determined up to isotopy. Otherwise, for  $k \neq 0, 4$ , depending on the positive local coordinate system, we obtain two non-isotopic embedded framed sphere  $\mathbb{S}_i$  and  $\bar{\mathbb{S}}_i$ . Note that for any framed sphere  $\mathbb{S} : S^{k-1} \times D^{4-k} \rightarrow M - T$  in a tangle  $(M, T)$ ,  $\bar{\mathbb{S}}$  is defined by

$$\bar{\mathbb{S}}(x, y) = \mathbb{S}(r_k(x), r_{4-k}(y)),$$

for  $x \in S^{k-1} \subset \mathbb{R}^k$ ,  $y \in D^{4-k} \subset \mathbb{R}^{4-k}$ , where

$$r_k(x_1, x_2, \dots, x_k) = (-x_1, x_2, \dots, x_k).$$

Moreover, we obtain a diffeomorphism

$$d_i : (M_{i-1}(\mathbb{S}_i), T_{i-1}) \longrightarrow (M_i, T_i)$$

which is given using the flow of the vector field  $\xi$  over  $M_{i-1} \setminus \text{nd}(\mathbb{S}_i)$ , the complement of the framed sphere  $\mathbb{S}_i$ . On the other hand, if  $g$  has index one critical points  $p_1, \dots, p_{n_i} \in G^{-1}[b_{i-1}, b_i] \cap F$ , the descending manifolds of  $p_1, \dots, p_{n_i}$  under the flow of  $\xi$  determines a set of orientation preserving framed arcs  $\mathbb{I}_i = \{\mathbb{I}_1^i, \dots, \mathbb{I}_{n_i}^i\}$  with end points on  $T_{i-1} = g^{-1}(b_{i-1})$ . Since  $g^{-1}(b_i)$  is a union of arcs connecting  $\partial^- M_i$  to  $\partial^+ M_i$ , the set  $\mathbb{I}_i$  is acceptable. The flow of  $\xi$  gives a diffeomorphism

$$d_i : (M_{i-1}, T_{i-1}(\mathbb{I}_i)) \longrightarrow (M_i, T_i).$$

Furthermore, if  $G$  has no critical points in  $\mathcal{W}_i$ , we set  $\mathbb{S}_i = \emptyset$  and the flow of  $\xi$  defines a diffeomorphism  $d_i$  from  $(M_{i-1}, T_{i-1})$  to  $(M_i, T_i)$ .

Therefore, any good Morse datum  $\mathfrak{M} = (G, \underline{b}, \xi)$  defines a well-defined parametrized Cerf decomposition for  $\mathcal{W}$  denoted by  $\mathfrak{C}(\mathfrak{M})$ , up to replacing some of the framed spheres  $\mathbb{S}$  by the corresponding framed spheres  $\bar{\mathbb{S}}$ .

**Lemma 4.6.** *Let  $\mathcal{W}$  be a stable cobordism. Then every parametrized Cerf decomposition for  $\mathcal{W}$  is induced by a good Morse datum.*

**Proof.** For any  $i$ , the parametrization of  $\mathcal{W}_i$  either contains a framed sphere  $\mathbb{S}_i$  or an acceptable set of framed arcs  $\mathbb{I}_i$ . If it contains a framed sphere  $\mathbb{S}_i \neq \emptyset$ , then as in the proof of Lemma 2.14 in [Juh1] we can define a Morse function  $G'_i : W(\mathbb{S}_i) \rightarrow [0, 1]$  together with an embedded gradient-like vector field  $\xi'_i$  on  $\mathcal{W}(\mathbb{S}_i) = (W(\mathbb{S}_i), F(\mathbb{S}_i))$  such that  $G'_i$  has a single critical point, while  $g'_i = G'_i|_{F(\mathbb{S}_i)}$  has no critical points. Furthermore, the diffeomorphism induced on  $(M_{i-1}(\mathbb{S}_i), T_{i-1})$  by  $G'_i$  and  $\xi'_i$  as above is  $\text{Id}_{M_{i-1}(\mathbb{S}_i)}$ . If  $\mathbb{S}_i = \emptyset$ , then for the cobordism  $\mathcal{W}(\mathbb{S}_i) = (M_{i-1}, T_{i-1}) \times [0, 1]$  we let  $G'_i = \pi_2$  and  $\xi'_i = \partial_t$ . If  $\mathcal{W}_i$  is accompanied with an acceptable set of framed arcs  $\mathbb{I}_i = \{\mathbb{I}_1^i, \dots, \mathbb{I}_{n_i}^i\}$ , we define a Morse function  $G'_i : W(\mathbb{I}_i) \rightarrow [0, 1]$  together with an embedded gradient-like vector field  $\xi'_i$  as follows. Let  $N$  be a small open neighborhood of  $D^2 \times D^1 \subset \mathbb{R}^3$  and  $H = N \times D^1$  and  $B \subset H$  be defined as before. For any  $1 \leq j \leq n_i$  consider a small neighborhood  $N(\mathbb{I}_j^i)$  of  $\mathbb{I}_j^i$  in  $(M_{i-1}, T_{i-1})$ . Let

$$(W_{ij}, F_{ij}) := \frac{(N(\mathbb{I}_j^i), N(\mathbb{I}_j^i) \cap T_{i-1}) \times [0, 1] \amalg (H_j, B_j)}{\mathbb{I}_j^i(D^2 \times D^1) \times \{1\} \sim D^2 \times D^1 \times \{-1\} \subset H_j} \subset \mathcal{W}(\mathbb{I}_i).$$

Here  $(H_j, B_j)$  denotes a copy of the standard pair  $\mathcal{H} = (H, B)$  for  $j = 1, \dots, n_i$ . After smoothing the corners obtained from attaching the standard pairs,  $(W_{ij}, F_{ij})$  is again diffeomorphic with  $\mathcal{H}$ . Choose a diffeomorphism

$$\phi_j : (W_{ij}, F_{ij}) \longrightarrow \mathcal{H} = (H, B)$$

such that

$$\begin{aligned} \phi_j(N(\mathbb{I}_j^i) \times \{1\}, (N(\mathbb{I}_j^i) \cap T_{i-1}) \times \{1\}) &= (N \times \{1\}, (B \cap N) \times \{1\}) \quad \text{and} \\ \phi_j(N(\mathbb{I}_j^i) \times \{0\}, (N(\mathbb{I}_j^i) \cap T_{i-1}) \times \{0\}) &= (N \times \{-1\}, (B \cap N) \times \{-1\}). \end{aligned}$$

Moreover, for a sufficiently small  $\nu \in \mathbb{R}^+$ , and every  $t \in (0, 1)$ , if  $(x, y, z) \in N$  belong to the  $\nu$ -neighborhood of  $\partial N$ , then

$$\phi^{-1}(x, y, z, 2t - 1) \in M_i \times \{t\}.$$

We define  $G'_i$  by

$$G'_i(y) := \begin{cases} \pi_2(y) & \text{if } y \in M_i \times I - \sqcup_{j=1}^{n_i} W_{ij} \\ \frac{\pi_2(\phi_j(y)) + 1}{2} & \text{if } y \in W_{ij} \end{cases}$$

This is a smooth function, by construction. We define the embedded gradient-like vector field  $\xi'_i$  on  $\mathcal{W}(\mathbb{I}_i)$  by pulling back the vector field  $\frac{1}{2}\partial_t$  on  $H_j$  using  $\phi_j$  and extending this to the rest of  $M_{i-1} \times I$  using  $\partial_t$ . It is now straightforward to check that the Morse function  $G'_i$  together with the embedded gradient-like vector field  $\xi'_i$  induces the identity diffeomorphism on  $(M_{i-1}, T_{i-1}(\mathbb{I}_i))$ .

Let  $G_i = a_i \circ G'_i \circ D_i^{-1}$  where  $a_i : [0, 1] \rightarrow [b_{i-1}, b_i]$  is the diffeomorphism  $a_i(t) = (1 - t)b_{i-1} + tb_i$  and  $D_i$  is the corresponding diffeomorphism from  $\mathcal{W}(\mathbb{S}_i)$  or  $\mathcal{W}(\mathbb{I}_i)$  to  $\mathcal{W}_i$ , which is equal to identity on  $(M_{i-1}, T_{i-1})$  and to  $d_i$  on  $(M_{i-1}(\mathbb{S}_i), T_{i-1})$  or  $(M_{i-1}, T_{i-1}(\mathbb{I}_i))$ . We can modify  $G_i$  and  $\xi = (D_i)_*(\xi'_i)$  on a collar neighborhood of  $(M_i, T_i)$  such that they fit together to give a Morse function  $G$  and an embedded gradient-like vector field  $\xi$  on  $\mathcal{W}$ .  $\square$

**Lemma 4.7.** *If the good Morse data  $\mathfrak{M} = (G, \underline{b}, \xi)$  and  $\mathfrak{M}' = (G', \underline{b}', \xi')$  induce the same parametrized Cerf decomposition of the stable cobordism  $\mathcal{W} = (W, F)$  in the sense that the attaching spheres and arcs coincide for given local coordinates, there exist diffeomorphisms*

$$D : \mathcal{W} \rightarrow \mathcal{W} \quad \text{and} \quad \phi : \mathbb{R} \rightarrow \mathbb{R}$$

such that

- (1)  $\underline{b}' = \phi(\underline{b})$
- (2)  $G' = \phi \circ G \circ D^{-1}$
- (3) for some  $h \in C^\infty(W, \mathbb{R}_+)$  we have  $D_*\xi = h.\xi'$
- (4)  $D|_M = \text{Id}_M$  and  $D|_{M'} = \text{Id}_{M'}$ .

**Proof.** The proof is completely similar to the proof of Lemma 2.14 in [Juh1].  $\square$

**Proposition 4.8.** *For every stable cobordism  $\mathcal{W}$  there exists a good Morse datum  $\mathfrak{M} = (G, \underline{b}, \xi)$ , and therefore a parametrized Cerf decomposition.*

**Proof.** Let  $g : F \rightarrow [0, 1]$  be a Morse function satisfying

$$g|_T \equiv 0, \quad g|_{T'} \equiv 1, \quad \text{and} \quad g|_{F \cap \partial_h W} = \pi_2.$$

Since every connected component of  $F$  has non-empty intersection with either of  $T$  and  $T'$ , we may assume that  $g$  has no minimal or maximal



index critical point. Moreover, we may assume that it has the same value over all of its critical points. We extend  $g$  to a tubular neighborhood of  $F$  such that it has no critical point and then extend it to a Morse function  $G : W \rightarrow [0, 1]$ . By a small perturbation in a neighborhood of the critical points of  $G$ , we may assume that  $G$  has distinct values over its critical points and these latter values are distinct from the value of  $g$  at its critical points. Finally, by a small perturbation in a neighborhood of the critical points of  $g$  we can transform  $G$  into a proper Morse function on  $\mathcal{W}$ , such that for every  $p, q \in \text{Crit}(g)$  with  $G(p) < G(q)$ ,  $G$  has no critical points on  $W$  above the interval  $(G(p), G(q))$ . It is straightforward to show that there is an embedded gradient-like vector field for every Morse function on  $\mathcal{W}$ . Let  $\xi$  be such a gradient-like vector field for  $G$ . Choose the ordered set

$$\underline{b} = \{0 = b_0 < b_1 < \dots < b_m = 1\} \subset [0, 1]$$

of regular values for  $G$  such that for every  $i = 1, \dots, m$ ,  $\text{Crit}_W(G)$  has at most one element in  $G^{-1}[b_{i-1}, b_i]$ . Furthermore, there is exactly one  $1 \leq j \leq m$  such that  $\text{Crit}_W(G)$  does not intersect  $G^{-1}[b_{j-1}, b_j]$  while for every  $p \in \text{Crit}(g)$  we have  $g(p) \in (b_{j-1}, b_j)$ . Now  $(G, \underline{b}, \xi)$  is a good Morse datum for  $\mathcal{W}$  and thus it gives a parametrized Cerf decomposition for  $\mathcal{W}$ .  $\square$

**4.3. Cerf moves.** In this subsection, we describe certain *Cerf moves* on parametrized Morse data and see how the induced parametrized Cerf decompositions change under these moves. For this purpose, let  $G, G' : W \rightarrow [a, b]$  be proper Morse functions and  $\mathfrak{M} = (G, \underline{b}, \xi)$  and  $\mathfrak{M}' = (G', \underline{b}', \xi')$  be parametrized Morse data on  $\mathcal{W} = (W, F)$ .

**Critical point cancellation/creation.** The Morse data  $\mathfrak{M}$  and  $\mathfrak{M}'$  are related by a *critical point cancellation* if

- (1)  $G$  and  $G'$  are related by a critical point cancellation i.e. there exist a family of smooth functions  $\{G_t : W \rightarrow [a, b] | t \in [-1, 1]\}$  such that  $G_{-1} = G$ ,  $G_1 = G'$  and:
  - For  $t \in [-1, 1] - \{0\}$ ,  $G_t$  is a proper Morse function on  $\mathcal{W}$  and  $G_0$  has a death singularity at some  $p_o \in W$ ,
  - The family is an elementary death ('chemin élémentaire de mort' in the sense of Cerf [Ce, Section 2.3, p.71]) with support in a neighborhood  $U$  of  $p_o$ . More precisely,  $G_t$  does not depend on  $t$  outside  $U$  and for  $t \in (0, 1]$  it has no critical points in  $U$ . Furthermore, there exist local coordinates  $(x_1, \dots, x_4)$  around  $p_o$ , such that for  $t \in [-1, 1]$

$$G_t(x_1, \dots, x_4) = G_0(p_o) + x_1^3 + tx_1 - x_2^2 \dots - x_k^2 + x_{k+1}^2 + \dots + x_4^2$$

- (2) There is some  $j \in \{1, \dots, m-1\}$  such that  $b_j = G_0(p_o)$ ,  $\underline{b}' = \underline{b} - \{b_j\}$  while

$$b_{j-1} < G_0(p_o) - 2/3\sqrt{3} \quad \text{and} \quad b_{j+1} > G_0(p_o) + 2/3\sqrt{3}.$$

- (3) The stable and unstable submanifolds  $W^s(q)$  and  $W^u(p)$  are transverse and intersect in a single flow line, where  $p$  and  $q$  are the index  $k$  and  $k+1$  critical points of  $G$  which cancel at  $t = 0$ . Moreover, the neighborhood  $U$  is in fact a neighborhood of

$$(W^u(p) \cup W^s(q)) \cap G^{-1}[b_{j-1}, b_{j+1}],$$

and  $\xi'$  coincide with  $\xi$  outside  $U$ .

A *critical point creation* is the reverse of a critical point cancellation.

Let the Morse data  $\mathfrak{M}$  and  $\mathfrak{M}'$  be related by a critical point cancellation as above. Then in the induced parametrized Cerf decomposition  $\mathfrak{C}(\mathfrak{M})$  the sphere  $\mathbb{S}_{j+1}$  intersects the image under  $d_j$  of the belt sphere of the handle attached to  $\mathbb{S}_j$  in a single point. Moreover, the cobordism  $\mathcal{W}_j \cup \mathcal{W}_{j+1}$  is a product cobordism and for

$$\mathbb{S}'_{j+1} = d_j^{-1}(\mathbb{S}_{j+1}) \subset M_{j-1}(\mathbb{S}_j) - T_{j-1}$$

there is a diffeomorphism

$$\phi : (M_{j-1}, T_{j-1}) \rightarrow (M_{j-1}(\mathbb{S}_j)(\mathbb{S}'_{j+1}), T_{j-1})$$

which is unique up to isotopy and fixes  $(M_{j-1}, T_{j-1}) \cap (M_{j-1}(\mathbb{S}_j)(\mathbb{S}'_{j+1}), T_{j-1})$ .

**Lemma 4.9.** *With the above notation fixed, the induced parametrized Cerf decomposition  $\mathfrak{C}(\mathfrak{M}')$  is obtained from  $\mathfrak{C}(\mathfrak{M})$  as follows. For every  $i < j-1$ , the tangle  $(M'_i, T'_i)$ , the framed sphere  $\mathbb{S}'_{i+1}$  (or the framed arc  $\mathbb{I}'_{i+1}$ ) and the diffeomorphism  $d'_{i+1}$  are the same as  $(M_i, T_i)$ ,  $\mathbb{S}_{i+1}$  (or  $\mathbb{I}_{i+1}$ ) and  $d_{i+1}$ , respectively. For  $i \geq j$ ,  $(M'_i, T'_i)$ , the framed sphere  $\mathbb{S}'_{i+1}$  (or the framed arc  $\mathbb{I}'_{i+1}$ ) and  $d'_{i+1}$  are the same as  $(M_{i+1}, T_{i+1})$ ,  $\mathbb{S}_{i+2}$  (or  $\mathbb{I}_{i+2}$ ) and  $d_{i+2}$ , respectively. Finally,  $(M'_{j-1}, T'_{j-1}) = (M_{j-1}, T_{j-1})$ ,  $\mathbb{S}'_j = \emptyset$  and*

$$d'_j : (M'_{j-1}(\mathbb{S}'_j), T'_{j-1}) = (M_{j-1}, T_{j-1}) \longrightarrow (M'_j, T'_j) = (M_{j+1}, T_{j+1})$$

is isotopic to  $d_{j+1} \circ d_j^{\mathbb{S}'_{j+1}} \circ \phi$  where

$$d_j^{\mathbb{S}'_{j+1}} : (M_{j-1}(\mathbb{S}_j)(\mathbb{S}'_{j+1}), T_{j-1}) \longrightarrow (M_j(\mathbb{S}_{j+1}), T_j)$$

is the diffeomorphism induced by  $d_j$ .

**Proof.** See the proof of Lemma 2.15 in [Juh1].  $\square$

**Critical point switches.** The Morse data  $\mathfrak{M}$  and  $\mathfrak{M}'$  are related by a *critical point switch* if  $\xi = \xi'$ , while for some  $0 \leq j \leq m$ ,  $G$  is related to  $G'$  either by switching critical points above the interval  $(b_{j-1}, b_j)$  with the ones above the interval  $(b_j, b_{j+1})$ , or switching two critical points above some interval  $(b_{j-1}, b_j)$ . In the first case,  $\underline{b} - b_j = \underline{b}' - b'_j$  while in the second case  $\underline{b} = \underline{b}'$ . More precisely,  $G$  is connected to  $G'$  by a smooth family  $\{G_t : W \rightarrow [a, b] | t \in [-1, 1]\}$  of Morse functions which are proper for all but finitely many values of  $t$  in  $[-1, 1]$  and  $\text{Crit}_{\mathcal{W}}(G_t)$  is independent of  $t$ .

Furthermore, depending on the type of the critical point switch, the smooth family  $\{G_t\}$  is of one of the following types.

**Type I.** For  $p, q \in \text{Crit}_W(G)$  we have

$$b_{j-1} < G(p) < b_j < G(q) < b_{j+1}.$$

Then  $G_0(p) = G_0(q)$  while  $tG_t(p) > tG_t(q)$  for  $t \neq 0$ . The family  $\{G_t\}_{t \in [-1,1]}$  is an elementary upward or downward switch ('chemin élémentaire de croisement, ascendant or descendente' in the sense of Cerf [Ce, Chapter II, p.40]) in a neighborhood  $U$  of

$$\begin{aligned} W_p^s(q) &:= W^s(q) \cap G^{-1}([G(p), G(q)]) \quad \text{or} \\ W_q^u(p) &:= W^u(p) \cap G^{-1}([G(p), G(q)]). \end{aligned}$$

In particular,  $G_t$  is independent of  $t$  outside  $U$  while in an open neighborhood containing the critical points inside  $U$ ,  $G_t - G$  is constant.

**Type II.** For  $p \in \text{Crit}_W(G)$  and  $q_1, \dots, q_n \in \text{Crit}(g)$  we have

$$b_{j-1} < G(p) < b_j < G(q_1) < \dots < G(q_n) < b_{j+1}.$$

Then  $G_t(q_1) < \dots < G_t(q_n) < G_t(p)$  for some  $\delta > 0$  and every  $1 - \delta < t \leq 1$ . Furthermore, the family  $\{G_t\}_{t \in [-1,1]}$  is an elementary switch ('chemin élémentaire de croisement' in the sense of Cerf [Ce, Chapter II, p.40]) with support in a neighborhood  $U$  of

$$W_q^u(p) := W^u(p) \cap G^{-1}([G(p), G(q_n)])$$

and  $G_t|_{W-U}$  is independent of  $t$ .

**Type III.** For  $p, q \in \text{Crit}(g)$  such that  $b_{j-1} < g(p) < g(q) < b_j$ , the function  $g$  has no critical points above the interval  $(g(p), g(q))$ . In this case we have  $G_0(p) = G_0(q)$ , while  $tG_t(p) > tG_t(q)$  for  $t \neq 0$ . Moreover, for a neighborhood  $U$  of

$$W^s(q) \cap G^{-1}([G(p), G(q)]) \text{ or } W^u(p) \cap G^{-1}[G(p), G(q)],$$

$G_t$  is independent of  $t$  in  $W - U$ . Furthermore, in a neighborhood  $V \subset U$  of the critical points  $G_t - G$  is constant.

If  $\mathfrak{M} = (G, \underline{b}, \xi)$  and  $\mathfrak{M}' = (G', \underline{b}', \xi')$  are related by a critical point switch of type III, then it is straightforward to see that  $\mathfrak{C}(\mathfrak{M})$  and  $\mathfrak{C}(\mathfrak{M}')$  are the same. However, if they are related by a critical point switch of type I or II, then  $\mathfrak{C}(\mathfrak{M})$  and  $\mathfrak{C}(\mathfrak{M}')$  are related as in Lemma 2.16 of [Juh1]. Let us recall the statement of the aforementioned Lemma (with a small modification) for a critical point switch of type II.

**Lemma 4.10.** (*Critical point switch of type II*) *With the above notation fixed, if  $\mathfrak{M} = (G, \underline{b}, \xi)$  and  $\mathfrak{M}' = (G', \underline{b}', \xi')$  are related by a critical point switch of type II, then the parametrized elementary cobordism  $\mathcal{W}_i$  coincides with  $\mathcal{W}'_i$  for any  $i < j - 1$  or  $i > j$ . Moreover,*

- $\mathbb{I}_{j+1} \cap d_j(\{0\} \times S^{4-k}) = \emptyset$  where  $k$  denotes the index of the critical point  $p$  and  $\{0\} \times S^{4-k} \subset D^k \times D^{4-k}$  is the belt sphere of the attached  $k$ -handle,
- $d_j(\mathbb{I}'_j) = \mathbb{I}_{j+1}$  and  $d'_j(\mathbb{S}_j) = \mathbb{S}'_{j+1}$ ,
- The diagram

$$\begin{array}{ccc}
 (M_{j-1}(\mathbb{S}_j), T_{j-1}(\mathbb{I}'_j)) & \xrightarrow{(d_j)^{\mathbb{I}'_j}} & (M_j, T_j(\mathbb{I}_{j+1})) \\
 \downarrow (d'_j)^{\mathbb{S}_j} & & \downarrow d_{j+1} \\
 (M'_j(\mathbb{S}'_{j+1}), T'_j) & \xrightarrow{d'_{j+1}} & (M_{j+1}, T_{j+1}).
 \end{array}$$

where  $(d_j)^{\mathbb{I}'_j}$  and  $(d'_j)^{\mathbb{S}_j}$  are induced by  $d_j$  and  $d'_j$ , respectively.

**Proof.** See proof of Lemma 2.16 in [Juh1].  $\square$

**Isotopy on embedded gradient-like vector field.** We say that the good Morse data  $\mathfrak{M} = (G, \underline{b}, \xi)$  and  $\mathfrak{M}' = (G', \underline{b}', \xi')$  are related by doing isotopy on embedded gradient-like vector field if  $G = G'$  and  $\underline{b} = \underline{b}'$ .

**Lemma 4.11.** *If  $\mathfrak{M}$  is related to  $\mathfrak{M}'$  by doing isotopy on the embedded gradient-like vector field, then the induced parametrized Cerf decompositions, after possibly reversing some of the framed spheres, are related as follows. For every  $j \in \{1, \dots, m\}$ , there is an ambient isotopy  $\{\phi_t\}_{t \in [0,1]}$  of  $(M_{j-1}, T_{j-1})$  with  $\phi_0 = \text{Id}_{M_{j-1}}$ , such that*

- If  $G$  has one critical point on  $W_j$ , then

$$\phi_1(\mathbb{S}_j) = \mathbb{S}'_j \quad \text{and} \quad d'_j = d_j \circ (\phi'_1)^{-1}$$

where  $(\mathbb{S}_j, d_j)$  and  $(\mathbb{S}'_j, d'_j)$  are the induced parametrizations of  $(W_j, F_j)$  and  $(W'_j, F'_j)$  and

$$\phi'_1 : (M_{j-1}(\mathbb{S}_j), T_{j-1}) \rightarrow (M_{j-1}(\mathbb{S}'_j), T_{j-1})$$

is the diffeomorphism induced by  $\phi_1$ .

- If  $g$  has critical points in  $F_j$ , then

$$\phi_1(\mathbb{I}_j) = \mathbb{I}'_j \quad \text{and} \quad d'_j = d_j \circ (\phi'_1)^{-1}$$

where  $(\mathbb{I}_j, d_j)$  and  $(\mathbb{I}'_j, d'_j)$  are the induced parametrizations of  $(W_j, F_j)$  and  $(W'_j, F'_j)$  and

$$\phi'_1 : (M_{j-1}, T_{j-1}(\mathbb{I}_j)) \rightarrow (M_{j-1}, T_{j-1}(\mathbb{I}'_j))$$

is the diffeomorphism induced by  $\phi_1$ .

**Proof.** Remark 2.11 of [Juh1].  $\square$

**Adding/removing regular values.** We say that the good Morse data  $\mathfrak{M} = (G, \underline{b}, \xi)$  and  $\mathfrak{M}' = (G', \underline{b}', \xi')$  are related by *adding or removing regular values*, if  $G = G'$  and  $\xi = \xi'$ . Thus,  $\mathfrak{M}(\underline{b} \cup \underline{b}') = (G, \underline{b} \cup \underline{b}', \xi)$  is a Morse datum for  $\mathcal{W}$  obtained from  $\mathfrak{M}$  and  $\mathfrak{M}'$  by adding regular values. In this case, the induced parametrized Cerf decomposition,  $\mathfrak{C}(\mathfrak{M}(\underline{b} \cup \underline{b}'))$ , is obtained from  $\mathfrak{C}(\mathfrak{M})$  and  $\mathfrak{C}(\mathfrak{M}')$  by *splitting* product cobordisms and cobordisms parametrized by acceptable sets of framed arcs.

**Definition 4.12.** We say that a parametrized Cerf decomposition  $\mathfrak{C}'$  is obtained from  $\mathfrak{C}$  by a *splitting*, if there is some  $j$  such that for any  $i < j$  the parametrized elementary cobordism  $\mathcal{W}'_i$  coincides with  $\mathcal{W}_i$  while for any  $i > j + 1$  it coincides with  $\mathcal{W}_{i-1}$ . Furthermore, the cobordism  $\mathcal{W}_j$  splits as  $\mathcal{W}'_j \cup_{(M'_j, T'_j)} \mathcal{W}'_{j+1}$  in  $\mathfrak{C}'$ , such that one of  $\mathcal{W}'_j$  or  $\mathcal{W}'_{j+1}$ , say  $\mathcal{W}'_j$ , is either parametrized by  $(\mathbb{S}'_j = \emptyset, d'_j)$  or  $(\mathbb{I}'_j, d'_j)$ . If  $\mathcal{W}'_j$  is product then, depending on the types of  $\mathcal{W}'_{j+1}$  and  $\mathcal{W}_j$ , we have either

$$\begin{aligned} \mathbb{S}_j &= d'^{-1}_j(\mathbb{S}'_{j+1}) \quad \text{and} \quad d_j = d'_{j+1} \circ (d'_j)^{\mathbb{S}_j}, \quad \text{or} \\ \mathbb{I}_j &= d'^{-1}_j(\mathbb{I}'_{j+1}) \quad \text{and} \quad d_j = d'_{j+1} \circ (d'_j)^{\mathbb{I}_j}. \end{aligned}$$

If  $\mathcal{W}'_j$  is parametrized by  $(\mathbb{I}'_j, d'_j)$  then  $\mathcal{W}'_{j+1}$  is parametrized by  $(\mathbb{I}'_{j+1}, d'_{j+1})$ . Moreover,

$$\mathbb{I}_j = \mathbb{I}'_j \sqcup d'^{-1}_j(\mathbb{I}'_{j+1}) \quad \text{and} \quad d_j = d'_{j+1} \circ (d'_j)^{\mathbb{I}_j - \mathbb{I}'_j}.$$

The reverse of this move, is called *merging*.

Therefore, for any two good Morse data  $\mathfrak{M} = (G, \underline{b}, \xi)$  and  $\mathfrak{M}' = (G, \underline{b}', \xi)$  we may change  $\mathfrak{C}(\mathfrak{M})$  to  $\mathfrak{C}(\mathfrak{M}')$  by first splitting and then merging.

**Left-right equivalence.** Let  $\mathcal{W} = (W, F)$  be a cobordism from the tangle  $(M, T)$  to the tangle  $(M', T')$ . We say that the Morse functions  $G$  and  $G'$  on  $\mathcal{W}$  are related by a *left-right equivalence* if there are diffeomorphisms  $\Phi : W \rightarrow W$  and  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  such that  $\Phi$  preserves  $(M, T)$  and  $(M', T')$ ,  $\Phi|_{(M, T)}$  and  $\Phi|_{(M', T')}$  are isotopic to  $\text{Id}_{(M, T)}$  and  $\text{Id}_{(M', T')}$ , respectively, and  $G' = \phi \circ G \circ \Phi^{-1}$ . Moreover, we say that the good Morse data  $\mathfrak{M} = (G, \underline{b}, \xi)$  and  $\mathfrak{M}' = (G', \underline{b}', \xi')$  are related by a *left-right equivalence*, if  $G$  and  $G'$  are related by a left-right equivalence and under the corresponding diffeomorphisms  $\underline{b}' = \phi \circ \underline{b}$  and  $\xi' = \Phi_*(\xi)$ . In this case, the parametrized Cerf decomposition  $\mathfrak{C}(\mathfrak{M}')$  is obtained from  $\mathfrak{C}(\mathfrak{M})$  by a *diffeomorphism equivalence*. This means that  $\mathcal{W}'_i = \Phi(\mathcal{W}_i)$  as parametrized elementary cobordisms i.e. depending on the type of  $\mathcal{W}_i$ ,  $\mathbb{S}'_i = \Phi(\mathbb{S}_i)$  or  $\mathbb{I}'_i = \Phi(\mathbb{I}_i)$  and  $d'_i = \Phi_i \circ d_i \circ (\Phi'_{i-1})^{-1}$ , where  $\Phi_i = \Phi|_{(M_i, T_i)}$  and  $\Phi'_i$  is the map induced by  $\Phi_i$  on  $(M_i(\mathbb{S}_{i+1}), T_i)$  or on  $(M_i, T_i(\mathbb{I}_{i+1}))$ .

**4.4. Parametrized Cerf decomposition theorem.** In this subsection, we show that any two good Morse data associated with a stable cobordism can be related by a sequence of Cerf moves. Let  $G, G' : W \rightarrow [a, b]$  be proper

Morse functions on  $\mathcal{W}$  together with the good Morse data  $\mathfrak{M} = (G, \underline{b}, \xi)$  and  $\mathfrak{M}' = (G', \underline{b}', \xi')$ .

**Lemma 4.13.** *With the above notation fixed, if there exist a smooth family  $\{G_t\}_{t \in [0,1]}$  of proper Morse functions on  $\mathcal{W}$  with  $G_0 = G$  and  $G_1 = G'$  then  $G$  is related to  $G'$  by a left-right equivalence.*

**Proof.** The proof is similar to the proof of Lemma 3.1 in [GWW] with minor modifications. For every  $t \in [0, 1]$ ,  $G_t$  is proper. Therefore, we have a smooth family  $\{\phi_t : \mathbb{R} \rightarrow \mathbb{R}\}_{t \in [0,1]}$  of diffeomorphisms of  $\mathbb{R}$  such that  $\phi_0 = \text{Id}$  and for every  $t \in [0, 1]$ ,  $\phi_t^{-1} \circ G_t$  has the same critical values as  $G$ . Moreover, if for  $p \in \text{Crit}_{\mathcal{W}}(G)$  and  $p_t \in \text{Crit}_{\mathcal{W}}(G_t)$  we have  $\phi_t^{-1} G_t(p_t) = G(p)$  then either both  $p$  and  $p_t$  are critical points in  $W$  or both are critical points in  $F$ .

Let  $G'_t = \phi_t^{-1} \circ G_t$ . Consider a point  $(x_0, t_0) \in W \times [0, 1]$  such that  $x_0 \in \text{Crit}_W(G'_{t_0})$ . For sufficiently small values of  $\epsilon > 0$  and  $\delta > 0$ , we can find the local coordinates  $\theta_t : B_\epsilon \subset \mathbb{R}^4 \rightarrow W$  for  $t \in (t_0 - \delta, t_0 + \delta)$  such that  $G'_t \circ \theta_t$  takes the normal form

$$G'_t \circ \theta_t(x_1, \dots, x_4) = G'_{t_0}(x_0) + \sum \pm x_i^2.$$

Similarly, if  $x_0 \in \text{Crit}(g_t)$ , for sufficiently small  $\epsilon, \delta > 0$  and  $t \in (t_0 - \delta, t_0 + \delta)$  we can find local coordinates  $\theta_t : B_\epsilon \subset \mathbb{R}^4 \rightarrow W$  so that  $F$  is given by  $\{x_3 = x_4 = 0\}$  in these coordinates and

$$G'_t \circ \theta_t(x_1, \dots, x_4) = G'_{t_0}(x_0) \pm x_1^2 \pm x_2^2 + x_3.$$

Then, for any  $(x, t)$  in a neighborhood of  $(x_0, t_0)$  defined as above, let

$$v(x, t) := \left( \frac{d}{ds} \Big|_{s=0} \theta_{t+s}(\theta_t^{-1}(x)), 1 \right).$$

It is straightforward to show that  $\{G'_t\}_{t \in [0,1]}$  is constant along the flow lines of  $v(x, t)$  in this neighborhood. Consider a finite set of pairs  $\{(U_i, v_i) | i = 1, \dots, n\}$ , where each pair consists of an open neighborhoods  $U_i \subset W \times [0, 1]$  as above and the corresponding vector field  $v_i$ , such that

$$\{(x, t) \in W \times [0, 1] \mid x \in \text{Crit}_{\mathcal{W}}(G'_t)\} \subset \bigcup_{i=1}^n U_i.$$

Consider an open set  $U_0 \subset W \times I$  in the complement of the critical points such that  $\bigcup_{i=0}^n U_i$  covers  $W \times [0, 1]$ . Let

$$v_0(x, t) := (- (\partial_t G'_t)(dG'_t(\xi_t))^{-1} \xi_t, 1)$$

where  $\{\xi_t\}_{t \in [0,1]}$  is a smooth family of vector fields on  $W$  such that for every  $t \in [0, 1]$ ,  $\xi_t$  is an embedded gradient-like vector field for  $G'_t$ . Note that  $\{G'_t\}_{t \in [0,1]}$  remains constant along the flow lines of  $v_0$ . Thus, we may patch the above local vector fields and construct a global vector field  $v$  on  $W \times [0, 1]$  such that  $\{G'_t\}_{t \in [0,1]}$  remains constant along its flow lines. Hence,

$G'_1 \circ \Phi_1 = G$ , where  $\Phi_1$  is the time-one map of the flow of  $v$ . Therefore,  $\phi_1^{-1} \circ G' \circ \phi_1 = G$ .  $\square$

**Definition 4.14.** *Given a stable cobordism  $\mathcal{W} = (W, F)$ . A proper Morse function  $G$  on  $\mathcal{W}$  is called almost ordered if  $G$  is ordered as a Morse function on  $W$  (i.e. for any  $p, q \in \text{Crit}_W(G)$ ,  $\text{ind}(p) < \text{ind}(q)$  implies  $G(p) < G(q)$ ) and for every  $p \in \text{Crit}_W(G)$  with  $\text{ind}(p) < 2$ ,  $G(p)$  is smaller than the critical values of  $g$ , while for  $p \in \text{Crit}_W(G)$  with  $\text{ind}(p) > 2$ ,  $G(p)$  is greater than the critical values of  $g$ . The Morse function  $G$  on  $\mathcal{W}$  is called ordered if  $G$  is almost ordered and for every  $p \in \text{Crit}_W(G)$  with  $\text{ind}(p) = 2$ ,  $G(p)$  is greater than the critical values of  $g$ .*

*A good Morse datum  $\mathfrak{M} = (G, \underline{b}, \xi)$  for  $\mathcal{W}$  is called almost ordered if  $G$  is almost ordered and it is called ordered if  $G$  is ordered.*

**Lemma 4.15.** *Any good Morse datum  $\mathfrak{M} = (G, \underline{b}, \xi)$  for  $\mathcal{W} = (W, F)$  can be connected to an ordered good Morse datum  $\mathfrak{M}' = (G', \underline{b}', \xi')$  by a sequence of critical point switches and removing regular values such that on an open neighborhood  $U$  of  $F$  we have  $G'|_U = G|_U$ .*

**Proof.** In order to prove this lemma, it is enough to show that for every  $p, q \in \text{crit}_W(G)$  with wrong order,  $G(p) < G(q)$ , no critical value of  $G$  in  $(G(p), G(q))$  and exactly one  $b_j \in \underline{b}$  in  $(G(p), G(q))$ , the Morse datum  $\mathfrak{M} = (G, \underline{b}, \xi)$  can be connected to a Morse datum  $\mathfrak{M}' = (G', \underline{b}', \xi')$  with a critical point switch such that  $G'(q) < G'(p)$  and in a neighborhood  $U$  of  $F$ ,  $G|_U = G'|_U$ . Depending on the type of the critical points  $p$  and  $q$  one of the following may happen:

**Case 1.**  $p, q \in \text{Crit}_W(G)$  and  $\text{ind}(p) > \text{ind}(q)$ . By a dimension count one can show that for a generic embedded gradient-like vector field  $\xi$ ,

$$W^u(p) \cap W^s(q) \cap G^{-1}(b_j) = \emptyset.$$

Moreover, for some  $\delta > 0$ ,

$$W^u(p) \cap g^{-1}(G(p), G(q) + \delta) = \emptyset \text{ and } W^s(q) \cap g^{-1}(G(p) - \delta, G(q)) = \emptyset.$$

Therefore, we have an elementary switch supported in a neighborhood of

$$W^u(p) \cap G^{-1}(G(p), G(q) + \delta) \text{ or } W^s(q) \cap G^{-1}(G(p) - \delta, G(q)),$$

disjoint from  $F$ , which connects  $G$  to a proper Morse function  $G'$  on  $\mathcal{W}$  satisfying  $G'(q) < G'(p)$ . Furthermore, in a small neighborhood  $U$  of  $F$  we have  $G|_U = G'|_U$ . Let  $\xi' = \xi$  and pick  $\underline{b}'$  such that  $\underline{b}' - b'_j = \underline{b} - b_j$  and  $G'(q) < b'_j < G'(p)$ . Then the resulted parametrized Morse datum  $\mathfrak{M}' = (G', \underline{b}', \xi')$  satisfies the required conditions.

**Case 2.**  $p \in \text{Crit}(g)$ ,  $q \in \text{Crit}_W(G)$  and  $\text{ind}(q) \leq 1$ . Let

$$\{p_1, \dots, p_{n_j} = p\} \subset \text{Crit}(g)$$

be the set of critical points of  $g$  such that

$$b_{j-1} < g(p_1) < \dots < g(p_{n_j}) < b_j.$$

Similar to case (1), after changing  $\xi$  to a generic embedded gradient-like vector field and by a dimension count one can assume that

$$\left(\bigcup_{i=1}^{n_j} W^u(p_i)\right) \cap W^s(q) \cap G^{-1}(b_j) = \emptyset.$$

Moreover, for some  $\delta > 0$ ,

$$W^s(q) \cap g^{-1}(G(p_1) - \delta, G(q)) = \emptyset.$$

Thus, there is an elementary switch, supported in a neighborhood of

$$W^s(q) \cap G^{-1}(G(p_1) - \delta, G(q)),$$

disjoint from  $F$ , which corresponds to a critical point switch on  $G$  that changes the order of  $p$  and  $q$ . The rest of the argument is as in case 1 with no modification.

**Case 3.**  $q \in \text{Crit}(g)$ ,  $p \in \text{Crit}_W(G)$  and  $\text{ind}(p) \geq 2$ . This is similar to case 2.  $\square$

**Proposition 4.16.** *Let  $\mathfrak{M} = (G, \underline{b}, \xi)$  and  $\mathfrak{M}' = (G', \underline{b}', \xi')$  be ordered indefinite good Morse data for a cobordism  $\mathcal{W} = (W, F)$ . If on a small tubular neighborhood  $U$  of  $F$  we have  $G|_U = G'|_U$  and  $\xi|_U = \xi'|_U$ , then  $\mathfrak{M}$  can be connected to  $\mathfrak{M}'$  by a sequence of critical point switches, critical point cancellations/creations, adding/removing regular values and left-right equivalences such that it stays indefinite and almost ordered throughout.*

**Proof.** Suppose that  $G, G' : W \rightarrow [a, b]$ . First, we show that  $G$  can be connected to  $G'$  by a smooth, generic family  $\{G_t\}_{t \in [0, 1]}$  such that for all but finitely many values of  $t$  in  $[0, 1]$ ,  $G_t$  is a proper indefinite and almost ordered Morse function. Moreover, for every  $t \in [0, 1]$  we have  $G_t|_U = G|_U = G'|_U$ .

The argument is similar to the proof of Theorem 4.5 in [GK]. Since  $G$  and  $G'$  coincide on  $U$ , there is a generic family  $\{G_t\}_{t \in [0, 1]}$  connecting  $G$  to  $G'$  such that  $G_t|_U = G|_U = G'|_U$ . Associated with  $\{G_t\}_t$  consider a generic family of embedded gradient-like vector fields  $\{\xi_t\}_t$  connecting  $\xi$  to  $\xi'$  such that  $\xi_t|_U$  does not depend on  $t$ . If  $\{G_t\}_{t \in [0, 1]}$  is not indefinite i.e.  $G_t$  is not indefinite for some  $t \in [0, 1]$ , we consider  $r \in [a, b]$  such that an index zero critical point is born at time  $r$  in  $p_r \in W$ . Corresponding to this critical point, we have a path

$$P = \{p_t \in \text{Crit}_W(G_t) \mid t \in [r, s]\}$$

in  $W$  such that for every  $t \in (r, s)$ ,  $p_t$  is an index zero critical point of  $G_t$ . Moreover, at  $p_s$  the index zero critical point is canceled against an index one



critical point. Since  $W$  is connected, for every  $t \in (r, s)$  there is an index one critical point  $q_t$  of  $G_t$  which cancels  $p_t$ , i.e.  $W^s(q_t) \cap W^u(p_t)$  is a single flow line. Therefore, for some  $\delta > 0$  and an ordered sequence

$$(r = t_0 < t_1 < \dots < t_n = s),$$

we have paths of index one critical points

$$Q^i = \{q_t^i \in \text{Crit}_W(G_t) \mid t \in I_i\}, \quad i = 1, \dots, n,$$

corresponding to the overlapping intervals

$$I_1 = [r, t_1 + \delta), I_2 = (t_1 - \delta, t_2 + \delta), \dots, I_n = (t_{n-1} - \delta, s],$$

such that  $q_t^i$  cancels  $p_t$  for  $t \in I_i$  and  $i = 1, \dots, n$ . Moreover,  $q_r^1 = p_r$  and  $q_s^n = p_s$ . For every  $1 \leq i \leq n$ , let

$$W_p^s(Q^i) := \bigcup_{t \in I_i} (W^s(q_t^i) \cap G_t^{-1}[G_t(p_t), G_t(q_t)]).$$

Counting dimensions implies that for a generic family  $\{\xi_t\}$ ,  $W_p^s(Q^i)$  does not intersect  $F$ .

Thus we can use the *Unmerge Lemma* [GK, Lemma 4.6] to cancel  $P$  against  $Q^i$  on the non-overlapping parts of the intervals  $I_i$  for  $i = 2, \dots, n-1$ . Then we use either *Eye Death Lemma* [GK, Lemma 4.7] or *Swallowtail Death Lemma* [GK, Lemma 4.8] to cancel over overlaps and on the intervals  $I_1$  and  $I_n$ .

Next, we need to make the family  $\{G_t\}_t$  almost ordered. First, we modify it using generic homotopies through *cusp-fold* crossings to move all critical point creations before all critical point switches and to move all critical point cancellations after all critical point switches. See [GK, p.12] for the exact definitions. Figure 2 shows how

$$\{(t, G_t(p)) \in [0, 1] \times [a, b] \mid p \in \text{Crit}(G_t)\}$$

changes through this homotopy.

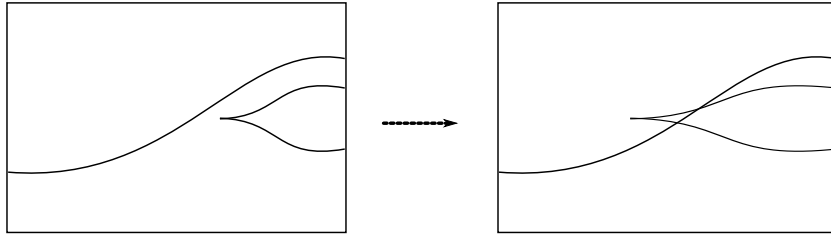


FIGURE 2. Cusp-fold crossing homotopy.

Second, we modify  $\{G_t\}$  such that if an index one/two (two/three) critical point creation/cancellation happens at point  $p_t \in W$  and time  $t$ , then  $G_t(p_t)$

is larger than the values of index one (two) critical points of  $G_t$  and smaller than the values of index two (three) critical points of  $G_t$ . Since  $G$  and  $G'$  are ordered, such modifications of  $\{G_t\}_{t \in [0,1]}$  can be achieved through generic homotopies whose support is disjoint from  $F$ .

Third, suppose that for paths of critical points

$$P = \{p_t \in \text{Crit}_W(G_t) \mid t \in I_{pq}\} \quad \text{and} \quad Q = \{q_t \in \text{Crit}_W(G_t) \mid t \in I_{pq}\}$$

with  $\text{ind}(P) < \text{ind}(Q)$  we have  $G_t(p_t) > G_t(q_t)$  for any  $t \in I_{pq}$ . Moreover, suppose  $G_t$  has no critical values in  $(G_t(p_t), G_t(q_t))$  for any  $t \in I_{pq}$ . We modify  $\{G_t\}$  using a generic homotopy whose support is disjoint from  $F$  and pull  $P$  below  $Q$  as follows. If we set

$$W_q^s(P) := \bigcup_{t \in I_{pq}} (W^s(p_t) \cap G_t^{-1}[G_t(q_t), G_t(p_t)]) \quad \text{and}$$

$$W_p^u(Q) := \bigcup_{t \in I_{pq}} (W^u(q_t) \cap G_t^{-1}[G_t(q_t), G_t(p_t)]).$$

it follows that

$$(W_q^s(P) \cap F) = \emptyset \quad \text{and} \quad (W_p^u(Q) \cap F) = \emptyset.$$

Furthermore, since  $\xi_t$  is generic, by counting dimensions we conclude that  $W_q^s(P)$  is disjoint from  $W_p^u(Q)$ . Therefore, there is a homotopy in a neighborhood of  $W_q^s(P)$  or  $W_p^u(Q)$  that switches  $P$  and  $Q$ .

Finally, let

$$P = \{p_t \in \text{Crit}_W(G) \mid t \in I_p\}$$

be a path of index one critical points on the interval  $I_p \subset [0, 1]$  such that  $g(q) < G_t(p_t) < g(q')$  for  $q, q' \in \text{Crit}(g)$  and  $g$  has no critical values between  $g(q)$  and  $g(q')$ . Moreover, assume that  $G_t$  has no critical values in  $(g(q), G_t(p_t))$  for  $t \in I_p$ . Since  $\xi_t$  is generic, by counting dimensions we conclude that  $W_{q_t}^s(P)$  and  $W_p^u(q_t)$  are disjoint. Moreover, since  $\xi_t$  is gradient-like for  $G_t$  as a function on  $\mathcal{W}$ , we have  $W_{q_t}^s(P) \cap F = \emptyset$ . Hence, we can change the homotopy in a neighborhood of  $W_{q_t}^s(P)$  and arrange for every  $t \in I_p$  that  $G_t(p_t) < g(q_t)$ .

Similarly, we may change  $\{G_t\}_t$  through homotopies such that for every index three critical point  $p_t$  of  $G_t$  (with  $t \in [0, 1]$ ) and every  $q \in \text{Crit}(g)$ , we have  $G_t(p_t) > g(q)$ .

As a result, we obtain a generic smooth path  $\{G_t\}_{t \in [0,1]}$  of Morse functions which are proper indefinite and almost ordered for all but finitely many values of  $t$ . Let  $m$  and  $M$  denote the minimum and the maximum of  $g$  over its critical points. Then, we need to arrange that there exist an ordered set

$$(0 = t_0 < t_1 < \dots < t_k = 1) \subset [0, 1]$$

$G_{t_i}$  is proper indefinite almost ordered with no critical point in  $W$  above the interval  $[m, M]$ . Moreover, such that for  $1 \leq i \leq k$  the family  $\{G_t\}_{t \in [t_{i-1}, t_i]}$  satisfies one of the following.

- (1) For every  $t \in [t_{i-1}, t_i]$ ,  $G_t$  is a proper Morse function on  $\mathcal{W}$ .
- (2)  $\{G_t\}_{t \in [t_{i-1}, t_i]}$  corresponds to a critical point creation/cancellation connecting  $G_{t_{i-1}}$  and  $G_{t_i}$ .
- (3)  $\{G_t\}_{t \in [t_{i-1}, t_i]}$  corresponds to switching two critical points of  $G_{t_{i-1}}$  on  $W$  with equal index, thus to a critical point switch of type I.
- (4)  $\{G_t\}_{t \in [t_{i-1}, t_i]}$  corresponds to switching some  $p \in \text{Crit}_W(G_{t_{i-1}})$  with the critical points of  $g$ , thus a critical point switch of type II.

In order to do so, we apply generic homotopies similar to Reidemeister II and Reidemeister III fold-crossings in [GK, p.11, p.12] [See Figures 3 and 4]. The only difference is that we also have paths consisting of the critical points of  $g$ . Let  $m$  and  $M$  denote the minimum and maximum value of  $g$  over its critical points. First, we apply Reidemeister III to make sure that if  $G_t(p) = G_t(p')$  for points  $p, p' \in \text{Crit}_W(G_t)$  with  $\text{ind}(p) = \text{ind}(p') = 2$  then either  $G_t(p) = G_t(p') < m$  or  $G_t(p) = G_t(p') > M$ . Using Reidemeister II we can then arrange for  $\{G_t\}_{t \in [0, 1]}$  to satisfy the required conditions. Note that the family  $\{G_t\}_{t \in [0, 1]}$  remains indefinite and almost ordered under these homotopies.

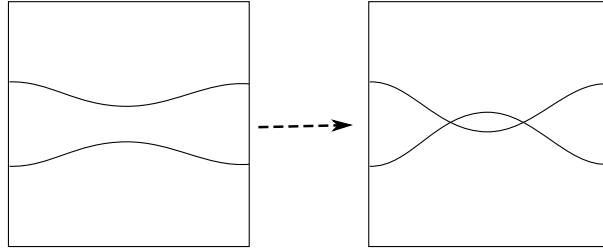


FIGURE 3. Reidemeister II fold-crossing.

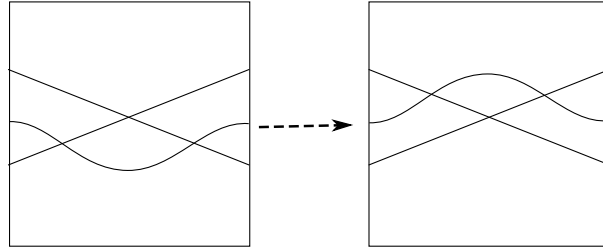


FIGURE 4. Reidemeister III fold-crossing.

Let us assume that an embedded gradient-like vector field  $\xi_i$ , together with an ordered set of regular values  $\underline{b}_i$  for  $G_{t_i}$  is given, such that  $\mathfrak{M}_i = (G_{t_i}, \xi_i, \underline{b}_i)$

is a good Morse datum for  $\mathcal{W}$  for some  $i \in \{0, 1, \dots, k-1\}$ . Moreover, assume that  $\underline{b}_i \cap [m, M] = \emptyset$ . Then, depending on the type of the family  $\{G_t\}_{t \in [t_i, t_{i+1}]}$  we may construct a good Morse datum  $\mathfrak{M}_{i+1} = (G_{t_{i+1}}, \xi_{i+1}, \underline{b}_{i+1})$ . In fact, if the family  $\{G_t\}_{t \in [t_i, t_{i+1}]}$  satisfies (1), (2), (3) or (4), then  $\mathfrak{M}_{i+1}$  is obtained from  $\mathfrak{M}_i$  by a left-right equivalence, a critical point creation/cancellation, a critical point switch of type I or a critical point switch of type II, respectively.

Therefore,  $\mathfrak{M}$  is related to a good Morse datum  $\mathfrak{M}_k = (G_k, \xi_k, \underline{b}_k)$ . Since  $G_k = G'$ ,  $\mathfrak{M}_k$  is related to  $\mathfrak{M}'$  by isotopies of the gradient-like vector field and adding or removing regular values. This completes the proof.  $\square$

**Lemma 4.17.** *Let  $\mathcal{W} = (W, F)$  be a stable cobordism and suppose that  $g, g' : F \rightarrow [a, b]$  are indefinite Morse functions over  $F$ . Then  $g$  can be connected to  $g'$  by a generic family  $\{g_t\}_{t \in [0, 1]}$  of indefinite Morse functions.*

**Proof.** This is a corollary of Theorem 4.5 in [GK].  $\square$

**Proposition 4.18.** *Let  $\mathfrak{M} = (G, \underline{b}, \xi)$  be a good indefinite and ordered Morse datum for the stable cobordism  $\mathcal{W} = (W, F)$ . Fix a proper indefinite and ordered Morse function  $G'' : W \rightarrow [a'', b'']$  on  $\mathcal{W}$ . Then  $\mathfrak{M}$  can be connected to a Morse datum  $\mathfrak{M}' = (G', \underline{b}', \xi')$  by a sequence of moves, including removing regular values, critical point switches of type III and left-right equivalences, such that the Morse datum remains ordered throughout and  $G'$  coincides with  $G''$  in a neighborhood of  $F$ .*

**Proof.** After removing some regular values, we may assume that for some integer  $n$ , all critical values of  $g$  lie in the interval  $(b_n, b_{n+1})$ . Choose the regular values  $m$  and  $M$  for  $G''$  such that

$$\forall p \in \text{Crit}_W(G'') \quad \begin{cases} G''(p) < m & \text{if } \text{ind}(p) = 1 \\ G''(p) > M & \text{if } \text{ind}(p) = 2 \end{cases}.$$

Moreover, we require that every critical point  $p \in F$  of  $g'' = G''|_F$  satisfies  $m < G''(p) < M$ . Let  $a = \min(\underline{b})$  and  $b = \max(\underline{b})$ . Consider a diffeomorphism  $\phi : [a, b] \rightarrow [a'', b'']$  with

$$\phi(a) = a'', \quad \phi(b) = b'', \quad \phi(b_n) = m \quad \text{and} \quad \phi(b_{n+1}) = M.$$

We apply the left-right equivalence move defined by  $\phi$  and  $\text{Id}_W$  on the Morse datum  $(G, \underline{b}, \xi)$  to get  $(\phi \circ G, \phi(\underline{b}), \xi)$ . Lemma 4.17 implies that  $\phi \circ g$  can be connected to  $g''$  by a generic family  $\{g_t\}_{t \in [0, 1]}$  of Morse functions on  $F$  which fails to be proper at the times

$$0 < c_1 < c_2 < \dots < c_l < 1.$$

Moreover, we may assume that for every  $t \in [0, 1]$  the critical values of  $g_t$  lie in the interval  $(m, M)$ . Consider a sufficiently small  $\delta_i > 0$  and let  $t_{2i-1} = c_i - \delta_i$  and  $t_{2i} = c_i + \delta_i$ . Therefore,

$$0 < t_1 < c_1 < t_2 < t_3 < c_2 < \dots < t_{2l-1} < c_l < t_{2l} < 1.$$

Let  $\tilde{\xi}_i$  be an embedded gradient-like vector field for  $g_i = g_{t_i}$ . For every  $i = 1, \dots, l$ , let  $p^i$  and  $q^i$  be the critical points of  $g_{c_i}$  such that

$$g_{c_i}(p^i) = g_{c_i}(q^i), \quad g_{2i-1}(p^i) < g_{2i-1}(q^i) \quad \text{and} \quad g_{2i}(p^i) > g_{2i}(q^i).$$

Since  $\delta_i$  is sufficiently small, we may perturb  $g_t$  for  $t \in [t_{2i-1}, t_{2i}]$  and change it to an elementary switch. Moreover, we may arrange for  $\tilde{\xi}_{2i-1} = \tilde{\xi}_{2i}$  such that the elementary switch  $\{g_t\}_{[t_{2i-1}, t_{2i}]}$  is supported in a neighborhood  $V_i$  of

$$W^u(p^i) \cap g_{2i-1}^{-1}[g_{2i-1}(p^i), g_{2i}(p^i)] \quad \text{or} \quad W^s(q^i) \cap g_{2i-1}^{-1}[g_{2i-1}(p^i), g_{2i}(p^i)].$$

Let  $G_{2i-1}$  be an extension of  $g_{2i-1}$  to an ordered Morse function on  $\mathcal{W}$ . Consider an embedded gradient-like vector field  $\xi_{2i-1}$  for  $G_{2i-1}$  such that  $\xi_{2i-1}|_F = \tilde{\xi}_{2i-1}$ . Without loss of generality we may assume that  $V_i$  is a neighborhood of

$$W^u(p^i) \cap g_{2i-1}^{-1}[g_{2i-1}(p^i), g_{2i}(p^i)].$$

Hence, for a bump function  $\omega_i$  supported in  $V_i$  we have

$$g_t = g_{2i-1} + (t - t_{2i-1})\omega_i \quad \text{for} \quad t \in [t_{2i-1}, t_{2i}].$$

Furthermore,  $\omega_i$  is constant in a neighborhood of  $p^i$ . We may extend  $\omega_i$  to a bump function on  $W$ , denote by  $\Omega_i$ , supported in an open neighborhood  $\overline{V}_i$  of

$$W^u(p^i) \cap G_{2i-1}^{-1}[G_{2i-1}(p^i), G_{2i}(p^i)]$$

such that for every  $t \in [t_{2i-1}, t_{2i}]$  the function

$$G_t = G_{2i-1} + (t - t_{2i-1})\Omega_i$$

is a Morse function on  $\mathcal{W}$  with  $\text{Crit}_{\mathcal{W}}(G_t) = \text{Crit}_{\mathcal{W}}(G_{2i-1})$ . The bump function  $\Omega_i$  is constant on an open neighborhood of  $p^i$  in  $W$ . Moreover,  $\xi_{2i-1}$  is an embedded gradient-like vector field for any  $G_t$ . Thus,  $\{G_t\}$  is an elementary switch for  $t \in [t_{2i-1}, t_{2i}]$ . For every  $t \in [t_{2i-1}, t_{2i}]$  we let  $\xi_t = \xi_{2i-1}$ .

On the other hand, we may find a family

$$\{\psi_t \mid t \in [t_{2i}, t_{2i+1}]\} \quad \text{for} \quad i = 1, \dots, l-1,$$

of diffeomorphisms of  $\mathbb{R}$  such that  $\psi_{t_{2i}} = \text{Id}$  and  $\psi_t \circ g_t$  have the same critical values as  $g_{2i}$ . With the same argument as in the proof of Lemma 4.13, we may define a vector field  $v(x, t)$  on  $F \times [t_{2i}, t_{2i+1}]$  such that the corresponding diffeomorphism

$$\Psi_t : F \times \{t_{2i}\} \rightarrow F \times \{t\}$$

satisfies

$$\psi_t \circ g_t \circ \Psi_t|_{F \times \{t_{2i}\}} = g_{2i}.$$

After choosing an arbitrary connection on a tubular neighborhood  $\text{nd}(F)$  of  $F$  in  $W$  we may extend  $v$  to a vector field over  $\text{nd}(F) \times [t_{2i}, t_{2i+1}]$  and correspondingly, extend  $\Psi_t$  from  $F \times \{t_{2i}\}$  to a diffeomorphism

$$\Psi_t : \text{nd}(F) \times \{t_{2i}\} \longrightarrow \text{nd}(F) \times \{t\}.$$

Suppose that we have an extension of  $g_{2i}$  to an ordered Morse function  $G_{2i}$  on  $\mathcal{W}$ . Then for any  $t \in [t_{2i}, t_{2i+1}]$  the Morse function  $G_{2i} \circ \Psi_t^{-1}$  is defined on the tubular neighborhood  $\text{nd}(F) \times \{t\} \subset W$  of  $F \times \{t\}$  and has no critical points. Fix the vector field  $\partial_t$  on the complement of

$$F \times [t_{2i}, t_{2i+1}] \subset W \times [t_{2i}, t_{2i+1}].$$

By patching the vector fields  $v$  and  $\partial_t$  using a partition of unity, we get a global vector field, still denoted by  $v(x, t)$ , on  $W \times [t_{2i}, t_{2i+1}]$ . The flow of  $v$  defines a family of diffeomorphisms

$$\{\Psi_t : W \times \{t_{2i}\} \rightarrow W \times \{t\} \mid t \in [t_{2i}, t_{2i+1}]\}.$$

Furthermore, the family

$$\{G_t := G_{2i} \circ \Psi_t^{-1} : W \times \{t\} \rightarrow \mathbb{R} \mid t \in [t_{2i}, t_{2i+1}]\}$$

is a family of ordered Morse function on  $\mathcal{W}$  such that  $G_t|_F = \phi_t \circ g_t$ . Therefore, any ordered Morse function  $G_{2i}$  on  $\mathcal{W}$  with  $G_{2i}|_F = g_{2i}$  can be connected by left-right equivalence to an ordered Morse function  $G_{2i+1}$  on  $\mathcal{W}$  such that  $G_{2i+1}|_F = g_{2i+1}$  and  $G_{2i+1}|_{W-\text{nd}(F)} = G_{2i}|_{W-\text{nd}(F)}$ .

We may thus connect  $\mathfrak{M} = (G, \xi, \underline{b})$  to a Morse datum  $\widetilde{\mathfrak{M}} = (\widetilde{G}, \widetilde{\xi}, \underline{b})$  by a sequence of left-right equivalences and critical point switches of type III, such that

$$\widetilde{G}|_F = G''|_F \quad \text{and} \quad \widetilde{G}|_{W-U} = G|_{W-U}$$

for an open neighborhood  $U$  of  $F$ . Moreover, the Morse datum remains good, indefinite and ordered throughout.

Finally, after a small perturbation of  $\widetilde{G}$  in the neighborhood of  $\text{Crit}(\widetilde{G}|_F)$  we may assume that  $t\widetilde{G} + (1-t)G''$  is a Morse function with no critical points in a tubular neighborhood  $\widetilde{U} \subset U$  of  $F$ . Let  $v'(x, t)$  be the corresponding vector field on  $\widetilde{U} \times [0, 1]$  as in the proof of Lemma 4.13. Define a global vector field  $v'$  on  $W \times [0, 1]$  by patching  $v'$  in  $\widetilde{U}$  with the vector field  $\partial_t = (0, 1)$  on  $(W - F) \times [0, 1]$  using a partition of unity. Denote the time-one flow of  $v'$  by  $\Phi_1$ . Then, the Morse datum  $\mathfrak{M}' = (G', \underline{b}', \xi')$  is obtained from  $\widetilde{\mathfrak{M}}$  by the left-right equivalence corresponding to  $\text{Id}_{\mathbb{R}}$  and  $\Phi_1^{-1}$  (so that in particular, we have  $G' = \widetilde{G} \circ \Phi_1^{-1}$ ), and satisfies the required conditions.  $\square$

Combining Proposition 4.16 and Proposition 4.18, one immediately arrives at the following theorem, which will be called the *parametrized Cerf decomposition theorem* in this paper.

**Theorem 4.19.** *Let  $\mathfrak{M} = (G, \underline{b}, \xi)$  and  $\mathfrak{M}' = (G', \underline{b}', \xi')$  be good indefinite Morse data for the stable cobordism  $\mathcal{W} = (W, F)$ . Then  $\mathfrak{M}$  and  $\mathfrak{M}'$  can be connected by a sequence of critical point switches, critical point creations/cancellations, isotopies of gradient-like vector field, adding/removing regular values and left-right equivalences. Moreover, we can avoid index zero and four critical points throughout the sequence.*

## 5. ONE-HANDLES, THREE-HANDLES AND THE COBORDISM MAPS

**5.1. Adding one-handles.** Let us assume that  $\mathbb{A}$  is an algebra over  $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$  as before and  $\mathcal{T} = [M, T, \mathfrak{s}, \mathbf{u}]$  is an  $\mathbb{A}$ -tangle. Let  $\mathbb{S}$  denote a framed 0-sphere in the complement of  $T$  in  $M$ . In other words,  $\mathbb{S}$  will be the union of two disjoint balls in  $M - T$ . The three-manifold  $M(\mathbb{S})$  is then obtained from  $M$  by adding a 1-handle. It will be diffeomorphic to  $M\#(S^1 \times S^2)$  if the disjoint balls are chosen in the same connected component of  $M - T$ . Otherwise, in the resulting 3-manifold  $M' = M(\mathbb{S})$ , one connected component corresponds to the connected sum of two connected components in  $M$ . Associated with the framed sphere  $\mathbb{S}$ , we also obtain the cobordism  $W(\mathbb{S})$  from  $M$  to  $M(\mathbb{S})$ , which is obtained from  $M \times [0, 1]$  by attaching a 1-handle along  $\mathbb{S} \times \{1\}$ . Let us assume that  $F = T \times [0, 1] \subset W(\mathbb{S})$  and regard  $\mathbf{u}$  also as a map  $\mathbf{u}_F : \pi_0(F) = \pi_0(T) \rightarrow \mathbb{A}$  or  $\mathbf{u}' : \pi_0(T') = \pi_0(T) \rightarrow \mathbb{A}$ , where  $T' = T \times \{1\} \subset M'$ . Every  $\text{Spin}^c$  class  $\mathfrak{t}$  on  $W(\mathbb{S})$  determines the  $\text{Spin}^c$  classes  $\mathfrak{s} = \mathfrak{t}|_M \in \text{Spin}^c(M)$  and  $\mathfrak{s}' = \mathfrak{t}|_{M'} \in \text{Spin}^c(M')$ . In this situation,  $\mathcal{C} = [W(\mathbb{S}), F, \mathfrak{t}, \mathbf{u}_F]$  is an  $\mathbb{A}$ -cobordism from  $\mathcal{T}$  to  $\mathcal{T}' = [M', T', \mathfrak{s}', \mathbf{u}']$ . In fact, the  $\text{Spin}^c$  structure  $\mathfrak{s}$  always determines  $\mathfrak{t}$ . Let

$$\mathcal{T}' = \mathcal{T}(\mathbb{S}) = [M(\mathbb{S}), T, \mathbf{u}, \mathfrak{s}'] \quad \text{and} \quad \mathcal{C} = \mathcal{C}(\mathbb{S}) = [W, F, \mathbf{u}_F, \mathfrak{t}].$$

In either case, we may choose a Heegaard surface  $\Sigma$  for  $M$  so that it cuts each one of the two disjoint balls in a disk. Denote the boundary curves of these two disks by  $C_1$  and  $C_2$ , and their centers by  $w_1$  and  $w_2$ , respectively. We may assume that each connected component  $T_i$  of  $T$  cuts  $\Sigma$  in a single transverse point  $z_i$  and that the collections  $\alpha$  and  $\beta$  of simple closed curves on  $\Sigma$ , which bound disks in  $M - T$ , give the  $\mathbb{A}$ -diagram

$$H = (\Sigma, \alpha, \beta, \mathbf{u} : \mathbf{z} = \{z_1, \dots, z_n\} \rightarrow \mathbb{A})$$

for  $\mathcal{T}$ .

In this situation, a natural  $\mathbb{A}$ -diagram for  $\mathcal{T}(\mathbb{S})$  may be constructed as follows. There is a properly embedded cylinder  $S$  in the one-handle attached to  $M$  with boundary circles  $C_1$  and  $C_2$ . If we remove the two disks with centers  $w_1$  and  $w_2$  from  $\Sigma$  and glue  $S$  to the resulting surface (which has  $C_1$  and  $C_2$  as its two boundary components) we obtain a Heegaard surface  $\Sigma'$  for  $M' = M(\mathbb{S})$ . Moreover, the union of the cylinder  $S$  and the aforementioned disks with centers  $w_1$  and  $w_2$  is a sphere, which will be denoted by  $\bar{S}$ . We may identify  $(\bar{S}, w_1, w_2)$  with  $(\mathbb{P}^1, 0, \infty)$ . Let us assume that  $\alpha$  and  $\beta$  are a pair of simple curves on  $S$  which bound disks on the two sides of  $\Sigma'$ , and cut each other in a pair of cancelling intersection points, i.e. are Hamiltonian isotopes of each other. We may label the two intersection points between  $\alpha$  and  $\beta$  by  $\theta_{\alpha\beta}$  and  $\theta_{\beta\alpha}$ , so that the bigons on the cylinder  $S$  connect  $\theta_{\alpha\beta}$  to  $\theta_{\beta\alpha}$  as the domains of the Whitney disks for  $(S, \alpha, \beta)$ . If the initial Heegaard diagram

$$H = (\Sigma, \alpha, \beta, \mathbf{u} : \mathbf{z} \rightarrow \mathbb{A}, \mathfrak{s})$$



is  $\mathfrak{s}$ -admissible in the strong sense of Remark 4.6 from [AE], the diagram

$$H' = (\Sigma', \alpha' = \alpha \cup \{\alpha\}, \beta' = \beta \cup \{\beta\}, \mathbf{u} : \mathbf{z} = \{z_1, \dots, z_n\} \rightarrow \mathbb{A}, \mathfrak{s}')$$

is an  $\mathbb{A}$ -diagram for  $\mathcal{T}'$ . We will thus assume the above stronger form of admissibility for the  $\mathbb{A}$ -diagram  $H$ .

With the above two admissible  $\mathbb{A}$ -diagrams fixed, and given an  $\mathbb{A}$ -module  $\mathbb{M}$ , we will construct a cobordism map from  $\mathrm{HF}^{\mathbb{M}}(\mathcal{T})$  to  $\mathrm{HF}^{\mathbb{M}}(\mathcal{T}')$  associated with adding the 1-handle along the framed 0-sphere  $\mathbb{S}$ . A completely similar construction would give the cobordism map associated with the 3-handles which are attached along framed 2-spheres.

**5.2. Stretching the necks in Heegaard diagrams.** The construction of the cobordism map for 1-handles rests on a slight generalization of Theorem 5.1 from [OS5], which will be discussed in the present subsection.

**Proposition 5.1.** *Fix the Heegaard diagrams*

$$H^i = (\Sigma^i, \alpha^i, \beta^i, \mathbf{w}^i = \{w_1^i, \dots, w_l^i\}), \quad i = 1, 2,$$

and let  $\Sigma$  denote the surface obtained from  $\Sigma^1$  and  $\Sigma^2$  by attaching  $l$  one-handles (necks) which connect  $w_j^1$  to  $w_j^2$ , for  $j = 1, \dots, l$ . Denote the number of curves in  $\alpha^i$  and  $\beta^i$  by  $d^i$  and the genus of  $\Sigma^i$  by  $g^i$ .

- Choose the class  $\phi^i$  of a Whitney disk for  $H^i$ ,  $i = 1, 2$ , with  $n_{w_j^1}(\phi^1) = n_{w_j^2}(\phi^2) = k_j$  and let  $\phi = \phi^1 \star \phi^2$  be a class obtained by joining  $\phi^1$  and  $\phi^2$  along the necks. Then

$$\mu(\phi) = \mu(\phi^1) + \mu(\phi^2) - 2(k_1 + \dots + k_l).$$

- Let  $J_{t_j}$  for  $j = 1, 2, \dots$  denote paths of almost complex structures associated with  $\Sigma$  which are stretched along the necks and converge to a degenerate path of almost complex structure on  $\Sigma^1 \vee \Sigma^2$ , giving the path  $J^i$  of almost complex structures on  $\Sigma^i$ . If  $\mathcal{M}(\phi)$  is non-empty for each  $J_{t_j}$  then the moduli spaces of broken pseudo-holomorphic flowlines representing  $\phi^1$  and  $\phi^2$  are non-empty.
- Let  $\mu(\phi^1) = 1$ ,  $\mu(\phi^2) = 2(k_1 + \dots + k_l)$  and suppose that  $\Sigma^2 \setminus \alpha^2$  and  $\Sigma^2 \setminus \beta^2$  only have components of genus zero, while  $d^2 > g^2$ . Consider

$$\rho^i : \mathcal{M}(\phi^i) \longrightarrow \mathrm{Sym}^{k_1}(\mathbb{D}) \times \dots \times \mathrm{Sym}^{k_l}(\mathbb{D}),$$

$$\rho^i(u) := \left( u^{-1} \left( \{w_j^i\} \times \mathrm{Sym}^{d_i-1} \right) \right)_{j=1}^l.$$

If the paths of almost complex structure  $J^1$  and  $J^2$  are generic, then the moduli space  $\mathcal{M}(\phi)$  may be identified with the fibered product

$$\begin{aligned} & \mathcal{M}(\phi^1) \times_{\mathrm{Sym}^{k_1}(\mathbb{D}) \times \dots \times \mathrm{Sym}^{k_l}(\mathbb{D})} \mathcal{M}(\phi^2) \\ &= \{(u^1, u^2) \mid u^i \in \mathcal{M}(\phi^i), \rho^1(u^1) = \rho^2(u^2)\} \end{aligned}$$

for sufficiently large values of  $j$ .

**Proof.** The proof of Proposition 5.1 is basically the same as the proof of Theorem 5.1 from [OS5], and uses the cylindrical reformulation of Heegaard Floer homology by Lipshitz [Lip]. We only need to make a small modification to the last part of the argument of Ozsváth and Szabó. In fact, the proof of the first two claims remains unchanged. We will thus focus on the proof of the last claim.

In Lipshitz reformulation, as we stretch the necks (i.e. as  $j \rightarrow \infty$ ), every sequence  $\{v_j\}_{j \in \mathbb{Z}^+}$  of curves with  $v_j \in \mathcal{M}_{J_{t_j}}(\phi)$  has a subsequence which converges to a pseudo holomorphic curve in the symplectic manifold

$$\begin{aligned} W(\infty) &= ((\Sigma^1 - \mathbf{w}^1) \times [0, 1] \times \mathbb{R}) \coprod ((\Sigma^2 - \mathbf{w}^2) \times [0, 1] \times \mathbb{R}) \\ &= (W^1 - \mathbf{w}^1 \times [0, 1] \times \mathbb{R}) \coprod (W^2 - \mathbf{w}^2 \times [0, 1] \times \mathbb{R}). \end{aligned}$$

which may be completed to a  $J^1$ -holomorphic curve in the symplectic manifold  $W^1 = \Sigma^1 \times [0, 1] \times \mathbb{R}$  and a  $J^2$ -holomorphic curve in  $W^2 = \Sigma^2 \times [0, 1] \times \mathbb{R}$ . The components of this limit consist of pre-glued flowlines, boundary degenerations and nodal curves supported entirely inside the fibers of the projection map to  $\mathbb{D} \simeq [0, 1] \times \mathbb{R}$ , which are identified with  $\Sigma^1 \vee \Sigma^2$ . By ignoring the matching conditions for the pre-glued flowlines we obtain the representative in  $\mathcal{M}(\phi^1)$  and  $\mathcal{M}(\phi^2)$  (this, in fact, proves the second claim).

Let us now assume that  $\mu(\phi^1) = 1$ . In the above Gromov limit, we obtain a  $J^1$ -holomorphic representative  $u^1$  of  $\phi^1$  and possibly a broken flow-line representative of  $\phi^2$ . If this latter representative has no closed components, a component  $u^2$  of it forms a pre-glued flowline together with  $u^1$ . Since other possible components of  $\phi^2$  (including the boundary degenerations) each correspond to a positive share of the Maslov index, it follows that  $\mu(u^2) \leq 2(k_1 + \dots + k_l)$ , with equality happening only if  $u^2$  represents  $\phi^2$  and there are no other components. For a generic point

$$\Delta = \rho^1(u^1) \in \text{Sym}^{k_1}(\mathbb{D}) \times \dots \times \text{Sym}^{k_l}(\mathbb{D})$$

the moduli space  $(\rho^2)^{-1}(\Delta)$ , which contains  $u^2$ , is of expected dimension  $\mu(u^2) - 2(k_1 + \dots + k_l)$ . Thus,  $u^2$  is forced to represent  $\phi^2$  if  $J^1$  (and thus  $\Delta$ ) is generic.

The difference with the argument of Ozsváth and Szabó appears when we consider the possibility of having closed components in the broken flow-line representing  $\phi^2$ . Let us assume that these closed components represent  $m[\Sigma^2]$ . After deleting these components we obtain a new class  $\psi^2$ , represented by  $u^2$ , with

$$\mu(\psi^2) = \mu(\phi^2) - 2m(d^2 - g^2 + 1).$$

Moreover, each component  $\Delta'_j$  of

$$\rho^2(u^2) = \Delta' = (\Delta'_1, \dots, \Delta'_l) \in \text{Sym}^{k_1}(\mathbb{D}) \times \dots \times \text{Sym}^{k_l}(\mathbb{D})$$

is obtained from the component  $\Delta_j$  of  $\rho^1(u^1) = \Delta$  by deleting  $m$  points  $\{p_1, \dots, p_m\}$  from it. Note that these  $m$  points are determined by the projection of the closed components in  $W(\infty) \times \mathbb{D}$  over  $\mathbb{D}$ , and are thus the same for  $\Delta_1, \dots, \Delta_l$ . This means that the components of  $\Delta$  have the points  $p_1, \dots, p_m$  in common. The subset of  $\mathcal{M}(\phi^1)$  consisting of  $J^1$ -holomorphic curves satisfying this condition is of expected dimension

$$\mu(\phi^1) - 2m(l-1) = 1 - 2m(l-1).$$

If  $m > 0$ ,  $l > 1$ , and  $J^1$  is generic, this subset of  $\mathcal{M}(\phi^1)$  is empty. The only remaining case is thus the case where  $l = 1$  and  $m > 0$ .

In this latter case, the argument of Ozsváth and Szabó may be used. The moduli space of all  $u^2 \in \mathcal{M}(\phi^2)$  with  $\rho^2(u^2) = \Delta'$  is of expected dimension

$$\begin{aligned} \mu(\phi_2) - 2(k_1 - m) &= 2k_1 - 2m(d_2 + 1 - g_2) - 2(k_1 - m) \\ &= -2m(d_2 - g_2) < 0 \end{aligned}$$

and hence is empty for a generic choice of  $J^1$  and  $J^2$ .

It follows that every weak limit of the curves in  $\mathcal{M}(\phi)$ , as we stretch the necks, is in correspondence with a pre-glued curve in

$$\begin{aligned} \{(u^1, u^2) \mid u^i \in \mathcal{M}(\phi^i), \rho^1(u^1) = \rho^2(u^2)\} \\ = \mathcal{M}(\phi^1) \times_{\text{Sym}^{k_1}(\mathbb{D}) \times \dots \times \text{Sym}^{k_l}(\mathbb{D})} \mathcal{M}(\phi^2). \end{aligned}$$

Moreover, from a pre-glued curve we may obtain an actual  $J_{t_j}$ -holomorphic curve if  $j$  is sufficiently large. This completes the proof.  $\square$

**5.3. Construction of the cobordism map.** The cobordism map is constructed from a chain map

$$\mathfrak{f}_{\mathbb{S}} : \text{CF}(\mathcal{T}) \longrightarrow \text{CF}(\mathcal{T}').$$

Choose a generic path  $J$  of almost complex structures associated with the surface  $\Sigma$ , and choose the generic path  $J'$  of almost complex structures associated with  $\Sigma'$  sufficiently close to the join of  $J$  and the standard complex structure  $J_S$  on the sphere  $\bar{S}$ , which is identified with  $\mathbb{P}^1$ . Corresponding to  $S$  we thus obtain the Heegaard diagram  $H_S = (\bar{S}, \alpha, \beta, \{w_1, w_2\})$ . Given a generator  $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$  of  $\text{CF}(\mathcal{T}, \mathfrak{s})$  (where this latter complex is defined using the Heegaard diagram  $H$  and the path  $J$  of almost complex structures), define

$$\mathfrak{f}_{\mathbb{S}}(\mathbf{x}) := \mathbf{x} \times \{\theta_{\alpha\beta}\} \in \text{CF}(\mathcal{T}'),$$

where  $\text{CF}(\mathcal{T}')$  is defined using the Heegaard diagram  $H'$  and the path  $J'$  of almost complex structures.

**Proposition 5.2.** *The homomorphism*

$$f_S : CF(\mathcal{T}) \longrightarrow CF(\mathcal{T}(\mathbb{S}))$$

*defined above is a chain map.*

**Proof.** Let us choose  $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$  so that it corresponds to the  $\text{Spin}^c$  class  $\mathfrak{s} \in \text{Spin}^c(M)$  and let

$$\mathbf{y} \times \theta \in \mathbb{T}_{\alpha'} \cap \mathbb{T}_{\beta'} = (\mathbb{T}_\alpha \cap \mathbb{T}_\beta) \times (\alpha \cap \beta)$$

denote a generator which contributes to  $df_S(\mathbf{x})$ , via  $u' \in \mathcal{M}(\phi')$ , where  $\phi' \in \pi_2(\mathbf{x} \times \theta_{\alpha\beta}, \mathbf{y} \times \theta)$  is the class of a Whitney disk with Maslov index 1. The class  $\phi'$  will be the join of the classes  $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$  (corresponding to the Heegaard diagram  $H$ ) and  $\phi_S \in \pi_2(\theta_{\alpha\beta}, \theta)$  (corresponding to the Heegaard diagram  $(\bar{S}, \alpha, \beta, \{w_1, w_2\})$ ), along a pair of necks corresponding to  $w_1$  and  $w_2$  which connect the two Heegaard diagrams.

Let  $|\theta|$  denote the number of intersection points in  $\theta$  which are different from the corresponding intersection points in  $\theta_{\alpha\beta}$ . Thus  $|\theta| \in \{0, 1\}$ . For a class  $\phi'$  as above with  $\mu(\phi') = 1$ , if  $u(\phi) \neq 0$  we will have

$$\begin{aligned} 1 &= \mu(\phi') \\ &= \mu(\phi) + \mu(\phi_S) - 2n_{w_1}(\phi_S) - 2n_{w_2}(\phi_S) \\ &= \mu(\phi) + |\theta|. \end{aligned}$$

If the path  $J'$  of almost complex structures is chosen close to the join of  $J$  and  $J_S$ , as described above, Proposition 5.1 implies that every pseudo-holomorphic representative  $u'$  of the class  $\phi'$  will be in correspondence with a  $J$ -holomorphic curve  $u \in \mathcal{M}(\phi)$  and a  $J_S$ -holomorphic curve  $u_S \in \mathcal{M}(\phi_S)$ , which are pre-glued.

If  $\theta \neq \theta_{\alpha\beta}$  then  $\mu(\phi) = 0$  and  $u$  is the constant map. It follows that  $n_{w_i}(\phi_S) = n_{w_i}(\phi) = 0$  and  $\phi_S$  corresponds to one of the two bigon connecting  $\theta_{\alpha\beta}$  to  $\theta_{\beta\alpha}$ . The total contribution of such  $u'$  to  $df_S(\mathbf{x})$  is thus zero.

The remaining contributions to  $df_S(\mathbf{x})$  come from the classes  $\phi'$  such that  $\mu(\phi) = 1$ , while  $\theta = \theta_{\alpha\beta}$  and  $n_{w_i}(\phi_S) = n_{w_i}(\phi) = k_i$  for  $i = 1, 2$ . We would like to show that the total contribution to  $df_S(\mathbf{x})$  from such  $u'$  is equal to the corresponding contribution from  $u \in \mathcal{M}(\phi)$  to  $d(\mathbf{x})$ .

Let us denote the union of all classes  $\phi_S \in \pi_2(\theta_{\alpha\beta}, \theta_{\alpha\beta})$  with  $n_{w_i}(\phi_S) = k_i$  for  $i = 1, 2$  by  $\phi_{k_1, k_2}$ . Consider the evaluation maps

$$\begin{aligned} \rho_\Sigma : \mathcal{M}(\phi) &\rightarrow \text{Sym}^{k_1}(\mathbb{D}) \times \text{Sym}^{k_2}(\mathbb{D}) \quad \text{and} \\ \rho_S : \mathcal{M}(\phi_{k_1, k_2}) &\rightarrow \text{Sym}^{k_1}(\mathbb{D}) \times \text{Sym}^{k_2}(\mathbb{D}). \end{aligned}$$

If  $u' \in \mathcal{M}(\phi \star \phi_S)$  correspond to the degeneration  $u \star u_S$  with  $u \in \mathcal{M}(\phi)$  and  $u_S \in \mathcal{M}(\phi_{k_1, k_2})$ , we further find

$$\rho_\Sigma(u) = \rho_S(u_S) \in \text{Sym}^{k_1}(\mathbb{D}) \times \text{Sym}^{k_2}(\mathbb{D}).$$

There are several classes  $\phi_S$  which form  $\phi_{k_1, k_2}$ . In fact such triangle classes are in correspondence with the pairs  $(a, b)$  of non-negative integers with  $a + b = k_1 + k_2$ . We thus obtain a map

$$\iota_\phi : \mathcal{M}(\phi') \simeq \mathcal{M}(\phi \star \phi_{k_1, k_2}) \longrightarrow \mathcal{M}(\phi) \times_{\text{Sym}^{k_1}(\mathbb{D}) \times \text{Sym}^{k_2}(\mathbb{D})} \mathcal{M}(\phi_{k_1, k_2}).$$

By Proposition 5.1, if the necks connecting  $S$  to  $\Sigma$  are sufficiently stretched and the paths of almost complex structures are generic  $\iota_\phi$  is a bijection, reducing the proof to the first part of the following lemma, which will be proved in Subsection 5.4. With this lemma in place, the proof of the proposition is complete.  $\square$

**Lemma 5.3.** *Let  $\alpha, \beta, \gamma$  denote three curves on  $\mathbb{P}^1$  which are small Hamiltonian isotopes of one another and denote the corresponding top intersection points by  $\theta_a \in \beta \cap \gamma, \theta_b \in \alpha \cap \gamma$  and  $\theta_c \in \alpha \cap \beta$ . Let  $w_1$  and  $w_2$  denote markings on the two domains in the complement of the isotopy regions. Let  $\phi_{k_1, k_2}$  denote the set of Whitney disks  $\phi \in \pi_2(\theta_c, \theta_c)$  with  $n_{w_i}(\phi) = k_i, i = 1, 2$  and  $\Delta_{k_1, k_2}$  denote the union of the triangle classes  $\Delta \in \pi_2(\theta_c, \theta_a, \theta_b)$  with  $n_{w_i}(\Delta) = k_i, i = 1, 2$ . Then for generic*

$$(p_1, p_2) \in \text{Sym}^{k_1}(\mathbb{D}) \times \text{Sym}^{k_2}(\mathbb{D})$$

and a generic path of almost complex structures on  $\mathbb{P}^1$  we have

- (i)  $n_{p_1, p_2}(\phi_{k_1, k_2}) := \# \{u \in \mathcal{M}(\phi_{k_1, k_2}) \mid \rho_{w_i}(u) = p_i, i = 1, 2\} = 1$  and
- (ii)  $n_{p_1, p_2}(\Delta_{k_1, k_2}) := \# \{u \in \mathcal{M}(\Delta_{k_1, k_2}) \mid \rho_{w_i}(u) = p_i, i = 1, 2\} = 1$ .

**5.4. Proof of Lemma 5.3.** The proof is closely related to the proof of Lemma 6.4 from [OS5]. We will prove the second claim. The first claim is in fact easier, and its proof is completely similar. The first step is to show that the number  $n_{p_1, p_2}(\Delta_{k_1, k_2})$  does not depend on the generic choice of  $(p_1, p_2)$ . Given a generic path  $\{(p_1^t, p_2^t)\}_{t \in [0, 1]}$  in  $\text{Sym}^{k_1}(\mathbb{D}) \times \text{Sym}^{k_2}(\mathbb{D})$  connecting the generic points  $(p_1^0, p_2^0)$  and  $(p_1^1, p_2^1)$ , consider the moduli space

$$\{(u, t) \mid t \in [0, 1], u \in \mathcal{M}(\Delta_{k_1, k_2}), \rho_{w_i}(u) = p_i^t, i = 1, 2\},$$

which is a smooth 1-dimensional moduli space with ends determined by the Gromov limits of its points. Picture  $\mathbb{D}$  as a  $Y$  shape domain, and assume that the complex structure is translation invariant as we move towards infinity in any of three directions corresponding to  $v_a, v_b$  and  $v_c$ . Three types of the boundary points correspond to a degenerations of the domain into a disk  $\phi$  and a triangle class  $\Delta_\phi$ . Since the path  $\{(p_1^t, p_2^t)\}_t$  remains in a compact subset of the domain, the disk  $\phi$  can not contain any of the pre-images of  $w_1$  or  $w_2$ , i.e.  $n_{w_1}(\phi) = n_{w_2}(\phi) = 0$ . There are no holomorphic disks to  $\theta_b$  with coefficient 0 at  $w_1$  and  $w_2$ . Thus,  $\phi$  corresponds to the disks contributing

to  $\widehat{\partial}(\theta_a)$  or  $\widehat{\partial}(\theta_c)$  which are all zero, i.e. the total number of such ends is always zero. The remaining ends correspond to

$$- \{u \in \mathcal{M}(\Delta_{k_1, k_2}) \mid \rho_{w_i}(u) = p_i^0, \ i = 1, 2\} \\ \coprod \{u \in \mathcal{M}(\Delta_{k_1, k_2}) \mid \rho_{w_i}(u) = p_i^1, \ i = 1, 2\},$$

implying the independence of  $n_{p_1, p_2}(\Delta_{k_1, k_2})$  from the generic choice of  $(p_1, p_2)$  in  $\text{Sym}^{k_1}(\mathbb{D}) \times \text{Sym}^{k_2}(\mathbb{D})$ .

Consider one of the branches of the  $Y$ -shape domain  $\mathbb{D}$  and identify the branch corresponding to  $v_a$  with  $[0, 1] \times (0, \infty)$ . Denote the projection over the second factor by  $\pi_{\mathbb{R}}$ . Choose the generic path  $(p_1^T, p_2^T)$  of points in  $\text{Sym}^{k_1}(\mathbb{D}) \times \text{Sym}^{k_2}(\mathbb{D})$  so that  $p_1^T$  is a union of  $k_1$  points  $p_{1,1}^T, \dots, p_{1,k_1}^T$  in the above branch so that

$$\pi_{\mathbb{R}}(p_{1,1}^T) > T \quad \text{and} \quad \pi_{\mathbb{R}}(p_{1,i+1}^T) - \pi_{\mathbb{R}}(p_{1,i}^T) > T.$$

Moreover,  $p_2^T$  is a union of  $k_2$  points  $p_{2,1}^T, \dots, p_{2,k_2}^T$  in the above branch so that

$$\pi_{\mathbb{R}}(p_{2,1}^T) - \pi_{\mathbb{R}}(p_{1,k_1}^T) > T \quad \text{and} \quad \pi_{\mathbb{R}}(p_{2,i+1}^T) - \pi_{\mathbb{R}}(p_{2,i}^T) > T.$$

One may then consider the ends of the smooth 1-dimensional moduli space

$$\coprod_{T \in [1, \infty)} \{u \in \mathcal{M}(\Delta_{k_1, k_2}) \mid \rho_{w_i}(u) = p_i^T, \ i = 1, 2\}$$

which is the union of

$$- \{u \in \mathcal{M}(\Delta_{k_1, k_2}) \mid \rho_{w_i}(u) = p_i^1, \ i = 1, 2\}$$

and the product

$$\mathcal{M}(\Delta_{0,0}) \times \left( \prod_{i=1}^{k_1} \{u \in \mathcal{M}(\phi) \mid \rho_{w_1}(u) = t_{1,i}\} \right) \\ \times \left( \prod_{i=1}^{k_2} \{u \in \mathcal{M}(\psi) \mid \rho_{w_1}(u) = t_{2,i}\} \right).$$

Here  $\phi \in \pi_2(\theta_a, \theta_a)$  denotes the homotopy types of the two disks connecting  $\theta_a$  to itself with boundary on  $\beta \sqcup \gamma$ ,  $n_{w_1}(\phi) = 1$  and  $n_{w_2}(\phi) = 0$ . Similarly,  $\psi \in \pi_2(\theta_a, \theta_a)$  denotes the homotopy types of the two disks connecting  $\theta_a$  to itself with boundary on  $\beta \sqcup \gamma$ ,  $n_{w_1}(\psi) = 0$  and  $n_{w_2}(\psi) = 1$ . Moreover,  $t_{1,i}$  and  $t_{2,i}$  are arbitrary point on  $[0, 1] \times \mathbb{R}$ . Lemma 7.3 from [AE] implies that the total number of points (counted with sign) in this latter end of the moduli space is 1. Consequently,  $n_{p_1^1, p_2^1}(\Delta_{k_1, k_2}) = 1$  and the proof is complete.

**5.5. Naturality of the cobordism map for one-handles.** In view of the discussion of naturality for tangle Floer homology in Section 3, we now prove:

**Theorem 5.4.** *Let  $\mathbb{A}$  denote a commutative algebra over  $\mathbb{Z}/2\mathbb{Z}$ ,  $\mathbb{M}$  denote an  $\mathbb{A}$ -module and  $\mathcal{T}$  denote an  $\mathbb{A}$ -tangle. Let*

$$\mathcal{C} = \mathcal{C}(\mathbb{S}) : \mathcal{T} \rightsquigarrow \mathcal{T}' = \mathcal{T}(\mathbb{S})$$

*denote the  $\mathbb{A}$ -cobordism obtained from the product cobordism  $\mathcal{T} \times [0, 1]$  by adding a 1-handle along the framed 0-sphere  $\mathbb{S}$ . Then the chain map  $\mathfrak{f}_{\mathcal{C}} = \mathfrak{f}_{\mathbb{S}}$  from Proposition 5.2 induces a natural homomorphism*

$$\mathfrak{f}_{\mathcal{C}} : \mathrm{HF}^{\mathbb{M}}(\mathcal{T}) \longrightarrow \mathrm{HF}^{\mathbb{M}}(\mathcal{T}').$$

**Proof.** The Heegaard surface  $\Sigma'$  for  $(M', T')$  cuts the sphere corresponding to the 1-handle in a simple closed curve. Every such surface in  $(M', T')$  determines the Heegaard surface  $\Sigma$  for  $M$  and a cylinder  $S$ , as a subsurface of  $\Sigma'$ . Moreover, we obtain the disk  $D_i$ ,  $i = 1, 2$  on  $\Sigma$  with center at a marked point  $w_i$ . Corresponding to any choice of the paths  $J$  of almost complex structures corresponding to  $\Sigma$  we obtain a degenerate path of almost complex structures corresponding to the join of the two surfaces  $\Sigma$  and the standard model  $\bar{S}$  at  $w_1$  and  $w_2$ . This path may be perturbed to a path  $J'$  of almost complex structures corresponding to  $\Sigma'$ . The independence of the isomorphism from the choice of these paths of almost complex structures is proved by standard techniques (involving a path connecting different choices of paths of almost complex structures). We will thus drop  $J$  and  $J'$  from the notation for simplicity.

Each  $\mathbb{A}$ -diagram  $H$  for  $\mathcal{T}$  corresponds to an isomorphism

$$\Phi_H : \mathrm{HF}^{\mathbb{M}}(H) \longrightarrow \mathrm{HF}^{\mathbb{M}}(M, T, \mathbf{u}, \mathfrak{s}),$$

and the Heegaard moves  $\mathfrak{h}$  changing  $H_0$  to  $H_1$  give the isomorphisms  $\Phi_{\mathfrak{h}}$  such that  $\Phi_{H_1} \circ \Phi_{\mathfrak{h}} = \Phi_{H_0}$ . Correspondingly, we obtain a sequence of moves  $\mathfrak{h}$  which changes the diagram  $H'_0 = H_0 \# H_S$  to  $H'_1 = H_1 \# H_S$ , where  $H_S$  is the simple diagram corresponding to the standard model  $\bar{S}$  in the above construction. Let us denote the chain map constructed in Proposition 5.2 from the diagram  $H_0$  by  $\mathfrak{f}_{H_0}$  and the corresponding map for  $H_1$  by  $\mathfrak{f}_{H_1}$ . We then need to show that the diagram

$$\begin{array}{ccc} \mathrm{HF}_{\mathbb{M}}(H_0, \mathfrak{t}|_M) & \xrightarrow{\mathfrak{f}_{H_0}} & \mathrm{HF}_{\mathbb{M}}(H'_0 = H_0 \# H_S, \mathfrak{t}|_{M'}) \\ \Phi_{\mathfrak{h}} \downarrow & & \downarrow \Phi_{\mathfrak{h}} \\ \mathrm{HF}_{\mathbb{M}}(H_1, \mathfrak{t}|_M) & \xrightarrow{\mathfrak{f}_{H_1}} & \mathrm{HF}_{\mathbb{M}}(H'_1 = H_1 \# H_S, \mathfrak{t}|_{M'}). \end{array}$$

is commutative. This is in turn reduced to the case where  $\mathfrak{h}$  consists of a single Heegaard move. When  $\mathfrak{h}$  is a stabilization or destabilization, the proof is straightforward, and the above diagram is in fact commutative in the level of chain complexes if the almost complex structure is chosen correctly. The remaining cases are thus the cases where  $\mathfrak{h}$  is an isotopy (which may pass over either of  $w_1$  and  $w_2$ ), or a handle slide.

We will present the argument in the case where  $\mathfrak{h}$  is a  $\beta$ -isotopy or a handle-slide on  $\beta$ . In this case,  $H'_1$  is obtained from  $H'_0$  by a Heegaard move, which may also be denoted by  $\mathfrak{h}$ . Let  $\gamma$  be a collection of curves which is obtained from  $\beta$  by applying the Heegaard move  $\mathfrak{h}$ , followed by small Hamiltonian isotopies. Choose a small isotope  $\gamma$  of  $\beta$  as well. Let  $\gamma' = \gamma \sqcup \{\gamma\}$ . There is a top generator associated with either of the Heegaard diagrams  $(\Sigma, \beta, \gamma, \mathbf{u}, \mathbf{s})$  and  $(\Sigma', \beta', \gamma', \mathbf{u}', \mathbf{s}')$ , which will be denoted by

$$\Theta = \Theta_{\beta\gamma} \quad \text{and} \quad \Theta' = \Theta_{\beta'\gamma'} = \Theta \times \{\theta_{\beta\gamma}\},$$

respectively. The Heegaard triple  $(\Sigma, \alpha, \beta, \gamma, \mathbf{u}, \mathbf{s})$  determines a holomorphic triangle, and together with the top generator  $\Theta$ , this gives a chain map

$$\mathfrak{f} = \mathfrak{f}_{\alpha\beta\gamma} : \text{CF}(\Sigma, \alpha, \beta, \mathbf{u}, \mathbf{s}) \longrightarrow \text{CF}(\Sigma, \alpha, \gamma, \mathbf{u}, \mathbf{s}).$$

Similarly, the Heegaard triple  $(\Sigma', \alpha', \beta', \gamma', \mathbf{u}', \mathbf{s}')$  and the top generator  $\Theta'$  determine a chain map

$$\mathfrak{f}' = \mathfrak{f}_{\alpha'\beta'\gamma'} : \text{CF}(\Sigma', \alpha', \beta', \mathbf{u}', \mathbf{s}') \longrightarrow \text{CF}(\Sigma', \alpha', \gamma', \mathbf{u}', \mathbf{s}').$$

To complete the proof, it is enough to show that  $\mathfrak{f}' \circ \mathfrak{f}_{H_0} = \mathfrak{f}_{H_1} \circ \mathfrak{f}$ .

For this purpose, fix the generator  $\mathbf{x}$  of  $\text{CF}(\Sigma, \alpha, \beta, \mathbf{u})$  and suppose

$$\Delta' \in \pi_2(\mathbf{x} \times \theta_{\alpha\beta}, \Theta \times \theta_{\beta\gamma}, \mathbf{y} \times \theta)$$

is a triangle class contributing to  $\mathfrak{f}' \circ \mathfrak{f}_{H_0}$ , where  $\theta$  is one of the two intersection points in  $\alpha \cap \gamma$ .  $\Delta'$  gives the triangle classes

$$\Delta \in \pi_2(\mathbf{x}, \Theta, \mathbf{y}) \quad \text{and} \quad \Delta_S \in \pi_2(\theta_{\alpha\beta}, \theta_{\beta\gamma}, \theta)$$

which support holomorphic representatives. Moreover we have

$$\begin{aligned} n_{w_1}(\Delta) &= n_{w_1}(\Delta_S), \quad n_{w_2}(\Delta) = n_{w_2}(\Delta_S) \quad \text{and} \\ 0 &= \mu(\Delta') = \mu(\Delta) + \mu(\Delta_S) - 2n_{w_1}(\Delta_S) - 2n_{w_2}(\Delta_S). \end{aligned}$$

Since  $\mu(\Delta_S) = |\theta| + 2n_{w_1}(\Delta_S) + 2n_{w_2}(\Delta_S)$  we find  $\mu(\Delta') = \mu(\Delta) + |\theta|$ . From here, we conclude that  $\theta$  is the top generator  $\theta_{\alpha\gamma}$  and that  $\mu(\Delta) = 0$ .

Let  $k_i = n_{w_i}(\Delta_S)$  for  $i = 1, 2$  and consider the evaluation maps

$$\begin{aligned} \rho_\Sigma : \mathcal{M}(\Delta) &\rightarrow \text{Sym}^{k_1}(\mathbb{D}) \times \text{Sym}^{k_2}(\mathbb{D}) \quad \text{and} \\ \rho_S : \mathcal{M}(\Delta_S) &\rightarrow \text{Sym}^{k_1}(\mathbb{D}) \times \text{Sym}^{k_2}(\mathbb{D}). \end{aligned}$$



If  $u' \in \mathcal{M}(\Delta')$  corresponds to the degeneration  $u \star u_S$  with  $u \in \mathcal{M}(\Delta)$  and  $u_S \in \mathcal{M}(\Delta_S)$ , we further find

$$\rho_\Sigma(u) = \rho_S(u_S) \in \text{Sym}^{k_1}(\mathbb{D}) \times \text{Sym}^{k_2}(\mathbb{D}).$$

There are several classes  $\Delta_S$  with the property that  $n_{w_i}(\Delta_S) = k_i$ . Let  $\Delta_{k_1, k_2}$  denote the set of all the above classes. We thus obtain a map

$$\iota_{\Delta'} : \mathcal{M}(\Delta') \longrightarrow \mathcal{M}(\Delta) \times_{\text{Sym}^{k_1}(\mathbb{D}) \times \text{Sym}^{k_2}(\mathbb{D})} \mathcal{M}(\Delta_{k_1, k_2}).$$

By the argument of Proposition 5.1, if the neck is sufficiently stretched and the paths of almost complex structures are generic  $\iota_{\Delta'}$  is a bijection, reducing the proof to the second claim in Lemma 5.3. This completes the proof of Theorem 5.4.  $\square$

**5.6. Three-handles.** Let us now assume that  $\mathbb{S} \subset M - T$  is a framed 2-sphere for the  $\mathbb{A}$ -tangle  $\mathcal{T} = [M, T, \mathbf{u}, \mathfrak{s}]$  which determines the cobordism

$$\mathcal{C}(\mathbb{S}) = [W(\mathbb{S}), F(\mathbb{S}) = T \times [0, 1], \mathbf{u}_F : \pi_0(T) = \pi_0(F(\mathbb{S})) \rightarrow \mathbb{A}, \mathfrak{t} = \mathfrak{t}_{\mathfrak{s}}],$$

where  $W(\mathbb{S})$  is obtained from  $M \times [0, 1]$  by adding a 3-handle along the framed sphere  $\mathbb{S} \times \{1\} \subset M \times \{1\}$ . We require that  $\langle c_1(\mathfrak{s}), [\mathbb{S}] \rangle = 0$ , and then the  $\text{Spin}^c$  structure  $\mathfrak{t}$  over  $W = W(\mathbb{S})$  is determined by  $\mathfrak{s}$ . The cobordism  $\mathcal{C}(\mathbb{S})$  connects  $\mathcal{T}$  to

$$\mathcal{T}' = \mathcal{T}(\mathbb{S}) = [M' = M(\mathbb{S}), T' = T \times \{1\}, \mathbf{u}' : \pi_0(T') = \pi_0(T) \rightarrow \mathbb{A}, \mathfrak{s}'].$$

If we reverse the cobordism  $\mathcal{C}(\mathbb{S})$  we obtain an  $\mathbb{A}$ -cobordism from  $\mathcal{T}'$  to  $\mathcal{T}$  which corresponds to the addition of a 1-handle.

A Heegaard surface  $\Sigma$  for  $M$  cuts the framed sphere  $\mathbb{S}$  in a cylinder  $S \subset \Sigma$  with boundary components  $C_1$  and  $C_2$ . A Heegaard surface  $\Sigma'$  for  $M'$  is then obtained from  $\Sigma$  by removing  $S$ , and gluing a pair of disks with centers  $w_1$  and  $w_2$  to  $C_1$  and  $C_2$ , respectively. If  $H' = (\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}', \mathbf{u}' : \mathbf{z} \rightarrow \mathbb{A}, \mathfrak{s}')$  is an  $\mathbb{A}$ -diagram for  $\mathcal{T}'$  and  $\alpha$  and  $\beta$  are a pair of Hamiltonian isotopic curves which are both isotopic to the core of the cylinder  $S$  and cut each other in  $\theta_{\alpha\beta}$  and  $\theta_{\beta\alpha}$ , then

$$H = (\Sigma, \boldsymbol{\alpha} = \boldsymbol{\alpha}' \cup \{\alpha\}, \boldsymbol{\beta} = \boldsymbol{\beta}' \cup \{\beta\}, \mathbf{u} : \mathbf{z} \rightarrow \mathbb{A}, \mathfrak{s})$$

will be an  $\mathbb{A}$ -diagram for  $\mathcal{T}$ , provided that we use the stronger form of admissibility for  $H'$ , as discussed in Remark 4.6 from [AE]. We thus obtain a chain map

$$\mathfrak{f}_{\mathbb{S}} : \text{CF}(\mathcal{T}) \longrightarrow \text{CF}(\mathcal{T}').$$

Choose a generic path  $J'$  of almost complex structures associated with the surface  $\Sigma'$ , and choose the generic path  $J$  of almost complex structures associated with  $\Sigma$  sufficiently close to the join of  $J'$  and the standard complex structure  $J_S$  on the sphere  $\bar{S}$  obtained from  $S$  by attaching a pair of disks

with centers  $w_1$  and  $w_2$  to  $S$ , so that  $(\overline{S}, w_1, w_2)$  is identified with  $(\mathbb{P}^1, 0, \infty)$ . Given a generator

$$\mathbf{x} = \mathbf{x}' \times \theta \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta = (\mathbb{T}_{\alpha'} \cap \mathbb{T}_{\beta'}) \times (\alpha \cap \beta)$$

of  $\text{CF}(\mathcal{T})$  (where this latter complex is defined using the Heegaard diagram  $H$  and the path  $J$  of almost complex structures), define  $\mathfrak{f}_S(\mathbf{x}) := \mathbf{x}'$  in  $\text{CF}(\mathcal{T}')$  if  $\theta = \theta_{\beta\alpha}$  is the intersection point with lower homological grading (the bottom intersection point), and define  $\mathfrak{f}_S(\mathbf{x}) = 0$ , otherwise. Here we assume that  $\text{CF}(\mathcal{T}')$  is defined using the Heegaard diagram  $H'$  and the path  $J'$  of almost complex structures.

An argument, which is basically the dual of the argument used for 1-handles implies the following theorem.

**Theorem 5.5.** *Let  $\mathbb{A}$  denote a commutative algebra over  $\mathbb{Z}/2\mathbb{Z}$ ,  $\mathbb{M}$  denote an  $\mathbb{A}$ -module and  $\mathcal{T} = [M, T, \mathfrak{s}, \mathfrak{u}]$  denote an  $\mathbb{A}$ -tangle. Let  $\mathbb{S} \subset M - T$  be a framed 2-sphere such that  $\langle c_1(\mathfrak{s}), [\mathbb{S}] \rangle = 0$  and*

$$\mathcal{C} = \mathcal{C}(\mathbb{S}) : \mathcal{T} \rightsquigarrow \mathcal{T}' = \mathcal{T}(\mathbb{S})$$

*denote the  $\mathbb{A}$ -cobordism which is obtained from the product  $\mathcal{T} \times [0, 1]$  by adding a 3-handle along  $\mathbb{S}$ . Then the map  $\mathfrak{f}_\mathcal{C} = \mathfrak{f}_\mathbb{S}$  defined above is a chain map and induces a natural homomorphism*

$$\mathfrak{f}_\mathcal{C}^\mathbb{M} : \text{HF}^\mathbb{M}(\mathcal{T}) \longrightarrow \text{HF}^\mathbb{M}(\mathcal{T}').$$

## 6. FRAMED ARCS, FRAMED KNOTS AND COBORDISM MAPS

**6.1. Special tangles corresponding to cobordisms.** Let assume that  $\mathcal{C} = [W, F, \mathfrak{t}, \mathfrak{u}_F]$  is an  $\mathbb{A}$ -cobordism from the  $\mathbb{A}$ -tangle  $\mathcal{T} = [M, T, \mathfrak{s}, \mathfrak{u}]$  to another  $\mathbb{A}$ -tangle  $\mathcal{T}' = [M', T', \mathfrak{s}', \mathfrak{u}']$ . In this subsection we introduce an  $\mathbb{A}$ -tangle  $\mathcal{T}_F$  to the  $\mathbb{A}$ -cobordism  $\mathcal{C}$  which plays the role of  $\#^k S^1 \times S^2$  in defining the chain maps in Heegaard Floer theory. The important feature of  $\mathcal{T}_F$  is the existence of a distinguished closed generator  $\Theta_F \in \text{CF}(\mathcal{T}_F)$ , which plays the role of the *top generator* in  $\text{HF}^-(\#^k S^1 \times S^2)$ .

Denote the stable cobordism  $[W, F]$  by  $\mathcal{W}$ . The positive boundary of  $\mathcal{W}$ , which is denote by

$$\partial^+ \mathcal{W} = (M^+, T^+) = (M^+(\mathcal{W}), T^+(\mathcal{W})),$$

may be regarded as a balanced tangle. In fact, under the identification

$$(M^+, T^+) = (\partial^+ M \times [0, 1], \partial^+ T \times [0, 1])$$

we may define

$$\begin{aligned} (\partial^+ M^+, \partial^+ T^+) &= (\partial^+ M \times \{1\}, \partial^+ T \times \{1\}) \quad \text{and} \\ (\partial^- M^+, \partial^- T^+) &= (\partial^+ M \times \{0\}, \partial^+ T \times \{0\}). \end{aligned}$$

Let  $J \subset F$  be a properly embedded simple arc such that  $\partial J \subset T^+$ . Associated with  $J$  we define a pair  $(M_J, T_J)$  by doing surgery on  $(M^+, T^+)$  at points  $J \cap T^+$  with the framing induced by  $F$ . More precisely, let  $\text{nd}(J) \subset W$  be a small tubular neighborhood of  $J$  in  $W$ . The intersection  $F \cap \partial \text{nd}(J)$  induces a framing on  $\partial \text{nd}(J)$ . Using this framing, we define

$$\begin{aligned} M_J &= (M^+ - M^+ \cap \text{nd}(J)) \cup \partial \text{nd}(J) \quad \text{and} \\ T_J &= \partial(F - \text{nd}(J)) - \partial F \cap (\partial W - M^+). \end{aligned}$$

If the end points of  $J$  are on distinct connected components of  $T^+$  then  $(M_J, T_J)$  is a balanced tangle. In this case, consider a small product neighborhood of  $M^+$ , and an identification of it with  $M^+ \times [0, \epsilon]$ , such that  $F \cap (M^+ \times [0, \epsilon]) = T^+ \times [0, \epsilon]$ . Denote the 4-manifold obtained from attaching the one-handle  $\text{nd}(J)$  to  $M^+ \times [0, \epsilon]$  by  $W_J$ . Let  $F_J = F \cap W_J$  and note that  $\mathcal{W}_J = (W_J, F_J)$  is a cobordism from  $(M^+, T^+)$  to  $(M_J, T_J)$ .

**Definition 6.1.** A set  $J = \{J_1, \dots, J_n\}$  of disjoint, properly embedded, simple arcs on  $F$  satisfying  $\partial J_i \subset T^+$  for  $i = 1, \dots, n$  is called a *spanning set* if each connected component of  $F - \coprod_{i=1}^n J_i$  is a disc and contains exactly one connected component of  $T$  and one connected component of  $T'$ .

Consider a spanning set  $J = \{J_1, \dots, J_n\}$  of arcs on  $F$ . After doing surgery on  $(M^+, T^+)$  along the elements of  $J$ , we get a balanced tangle, which is denoted by  $(M_J, T_J)$ . In particular, if  $J$  is a single arc,  $(M_{\{J\}}, T_{\{J\}}) = (M_J, T_J)$ .

**Lemma 6.2.** *The diffeomorphism type of the tangle  $(M_J, T_J)$  does not depend on the choice of  $J$ .*

**Proof.** Associated with a set  $J$  of disjoint, properly embedded, simple arcs with the above properties, we may define a Morse function  $g : F \rightarrow [0, 1]$  together with a gradient like vector field  $\xi$ , such that

- (1)  $g|_{F \cap \partial^- W} = 0$ ,  $g|_{F \cap \partial^+ W} = 1$  and  $g$  has no critical point in a neighborhood of  $\partial F$ .
- (2) All critical points of  $g$  have index one.
- (3) The arcs  $J_1, \dots, J_n$  are the unstable manifolds of the critical points of  $g$ .

Let  $J$  and  $J'$  be two sets of disjoint, properly embedded simple arcs on  $F$  satisfying the above conditions. Consider the Morse functions  $g$  and  $g'$  together with gradient-like vector fields  $\xi$  and  $\xi'$  associated with  $J$  and  $J'$ , respectively. Theorem 4.5 in [GK] implies that the Morse data  $(g, \xi)$  can be connected to  $(g', \xi')$  by a sequence of critical point switches and isotopies of the gradient-like vector field. Thus  $J$  can be connected to  $J'$  by a sequence of arc slides. Here, an arc slide of  $J_1$  over  $J_2$  is replacing  $J_1$  with the arc obtained by sliding one of the feet of  $J_1$  over  $J_2$ . As a result, since  $(M_J, T_J)$  does not change under arc slides, it is independent of  $J$ .  $\square$

As a consequence of the Lemma 6.2, we may denote the balanced tangle  $(M_J, T_J)$  by  $(M_F, T_F)$ . Similarly, the corresponding cobordism  $\mathcal{W}_J = (W_J, F_J)$  from  $(M^+, T^+)$  to  $(M_F, T_F)$  may be denoted by  $\mathcal{W}_F$ . Let  $H_i \in H_2(M_F, \mathbb{Z})$  denote the homology class represented by the belt sphere of the attached 1-handle corresponding to  $J_i$ . Consider the  $\text{Spin}^c$  class  $\mathfrak{s}_0 \in \text{Spin}^c(M_F)$  such that

$$\langle c_1(\mathfrak{s}_0), H_i \rangle = 0 \quad \forall i = 1, 2, \dots, m.$$

Let  $\mathcal{T}_F = [M_F, T_F, \mathfrak{s}_0, \mathbf{u}_F \circ \iota]$  denote the  $\mathbb{A}$ -tangle obtained from  $(M_F, T_F)$ , the  $\text{Spin}^c$  class  $\mathfrak{s}_0$  and the representation  $\mathbf{u}_F \circ \iota$ , where  $\iota : \pi_0((T_F)) \rightarrow \pi_0((F))$  is induced by inclusion. Let us assume that  $\partial^+ T = \{p_1, \dots, p_\kappa\} \subset \partial^+ M = S$  and correspondingly, the tangle  $T^+$  is a union of components

$$T^+ = \coprod_{i=1}^{\kappa} T_i^+ = \coprod_{i=1}^{\kappa} (\{p_i\} \times [0, 1]) \subset M^+.$$

For every  $1 \leq i \leq \kappa$  consider a point  $\bar{p}_i \in S$  close to  $p_i$  and let  $\bar{T}_i = \bar{p}_i \times [0, 1]$  and  $\bar{T} := \sqcup_{i=1}^{\kappa} \bar{T}_i \subset M^+$ . Denote the balanced tangle constructed by doing surgery on  $(M^+, T^+ \sqcup \bar{T})$  along a spanning set  $J$  of disjoint, properly embedded simple arcs in  $F$  by  $(M_F, T_F \sqcup \bar{T}_F)$ .

Suppose that  $F = \coprod_{i=1}^m F_i$ . Consider a variable  $\mathbf{u}_i$  associated with each component  $F_i$  of  $F$  and a variable  $\mathbf{v}_j$  associated with each component of  $\bar{T} = \coprod_{j=1}^{\kappa} \bar{T}_j$ . Furthermore, assume that  $\partial^+ M = S = \coprod_{k=1}^{\ell} S_k$  are the

connected components of  $\partial^+ M$ , while  $T_j$  is the connected component of  $T_F$  which includes  $(p_j, 0) \subset \partial^+ M \times [0, 1]$  for  $j = 1, \dots, \kappa$ . Define  $\mathbf{u}(p_j) = \mathbf{u}_i$  if  $p_j \in F_i$  and set

$$\mathbf{u}^k = \prod_{p_j \in S_k} (\mathbf{u}(p_j) \mathbf{v}_j) \in \mathbb{F}[\mathbf{u}_1, \dots, \mathbf{u}_m, \mathbf{v}_1, \dots, \mathbf{v}_\kappa].$$

Denote the genus of  $S_k$  by  $g_k$ . With this notation fixed, let

$$\mathbb{A}_F^+ := \frac{\mathbb{F}[\mathbf{u}_1, \dots, \mathbf{u}_m, \mathbf{v}_1, \dots, \mathbf{v}_\kappa]}{\langle \mathbf{u}^k \mid g_k > 0, k = 1, \dots, \ell \rangle}.$$

Define the representation  $\mathbf{u}_F^+ : \pi_0(T_F \sqcup \overline{T}_F) \rightarrow \mathbb{A}_F^+$  by

$$\begin{cases} \mathbf{u}_F^+(\overline{T}_j) = \mathbf{v}_j & 1 \leq j \leq \kappa \\ \mathbf{u}_F^+(T_j) = \mathbf{u}(p_j) & 1 \leq j \leq \kappa. \end{cases}$$

Thus  $\mathcal{T}^+ = [M_F, T_F \sqcup \overline{T}_F, \mathbf{u}_F^+, \mathfrak{s}_0]$  is an  $\mathbb{A}_F^+$ -tangle.

**6.2. The distinguished generator.** For every  $1 \leq i \leq \kappa$  consider a product disk  $(D_i, \partial D_i)$  in the balanced tangle  $(M^+, T^+ \sqcup \overline{T})$  such that

$$\overline{T}_i \sqcup -T_i^+ \subset \partial D_i \quad \text{and} \quad \partial D_i - (\overline{T}_i \sqcup -T_i^+) \subset \partial M^+.$$

Abusing the notation, we denote the intersection of  $D_i$  with  $M_F$  by  $D_i$  as well. Let  $D = \sqcup_{i=1}^\kappa D_i$ . For every  $1 \leq j \leq n$ , we may attach an oriented one handle  $D'_j$  (i.e. a band) to  $D$  such that  $D'_j \subset M_F$  is embedded in the one handle associated with  $J_j$  and

$$\partial D'_j - (\partial D'_j \cap D) \subset F.$$

Denote the resulted embedded, oriented surface in  $(M_F, T_F \sqcup \overline{T}_F)$  by  $F'$ . Note that

$$(F', \partial F') \subset (M_F, \partial M_F \cup (T_F \sqcup \overline{T}_F))$$

and that this pair is uniquely determined up to isotopy.

Let  $(X_F, \tau_F \sqcup \overline{\tau}_F)$  be the balanced sutured manifold associated with the tangle  $(M_F, T_F \sqcup \overline{T}_F)$ . The surface  $F' \cap X_F \subset X_F$  is then a decomposing surface in  $(X_F, \tau_F \sqcup \overline{\tau}_F)$  and it defines a sutured manifold decomposition

$$(X_F, \tau_F \sqcup \overline{\tau}_F) \rightsquigarrow^{F'} (X'_F, \tau'_F \sqcup \overline{\tau}'_F).$$

Note that  $(X'_F, \tau'_F \sqcup \overline{\tau}'_F)$  is a product sutured manifold. Thus, there exists a unique relative  $\text{Spin}^c$  structure  $\underline{\mathfrak{s}}_{F'} \in \text{Spin}^c(X_F, \tau_F \sqcup \overline{\tau}_F)$  which is outer with respect to  $F'$ . Recall that a relative  $\text{Spin}^c$  structure  $\underline{\mathfrak{s}} \in \text{Spin}^c(X_F, \tau_F \sqcup \overline{\tau}_F)$  is called outer with respect to  $F'$  if it is represented by a unit vector field  $v$  on  $X_F$  such that  $v_p \neq -(\nu_{F'})_p$  for every  $p \in F'$ . Here  $\nu_{F'}$  is the unit normal vector field of  $F'$  with respect to some Riemannian metric on  $X_F$ , See [Ju2, Definition 1.1]. Note that  $[\underline{\mathfrak{s}}_{F'}] = \mathfrak{s}_0$ .

For  $j_1, j_2 \in \{1, \dots, \kappa\}$  and  $i \in \{1, \dots, m\}$ , if  $T_{j_1}, T_{j_2} \subset \partial F_i$  then

$$[\tau_{j_1}] = [\tau_{j_2}] \in H_1(X_F, \mathbb{Z}).$$

Thus, there is a well-defined homology class  $h_i = [\tau_j] \in H_1(X_F, \mathbb{Z})$  for a suture  $\tau_j$  corresponding to  $T_j \subset \partial F_i$ . As a result, the algebra  $\mathbb{A}_F^+$  admits a filtration by  $\mathbb{H} = H^2(X_F, \partial X_F, \mathbb{Z})$ , which is defined as

$$\begin{aligned} \chi : G(\mathbb{A}_F^+) &\rightarrow \mathbb{H} = H^2(X_F, \partial X_F, \mathbb{Z}) \\ \chi\left(\prod_{i=1}^m u_i^{a_i} \prod_{j=1}^{\kappa} v_j^{b_j}\right) &:= \sum_{i=1}^m a_i \text{PD}(h_i) + \sum_{j=1}^{\kappa} b_j \text{PD}(\bar{\tau}_j) \end{aligned}$$

where  $\bar{\tau}_j \subset \bar{\tau}$  is the suture corresponding to  $\bar{T}_j \subset \bar{T}_F$ . Therefore, for every  $\mathfrak{s} \in \text{Spin}^c(M_F)$

$$\text{CF}(M_F, T_F \sqcup \bar{T}_F, \mathbf{u}_F^+, \mathfrak{s})$$

may be decomposed into sub-complexes associated with relative  $\text{Spin}^c$  structures. More precisely, let  $H = (\Sigma, \alpha, \beta, \mathbf{u}_F^+ : \mathbf{z} \sqcup \bar{\mathbf{z}} \rightarrow \mathbb{A}_F^+, \mathfrak{s}_0)$  be an  $\mathbb{A}$ -diagram for  $\mathcal{T}^+$ . Then

$$\text{CF}(\Sigma, \alpha, \beta, \mathbf{u}_F^+, \mathfrak{s}_0) = \bigoplus_{\mathfrak{s} \in \mathfrak{s}_0 \subset \text{Spin}^c(M_F, T_F \sqcup \bar{T}_F)} \text{CF}(\Sigma, \alpha, \beta, \mathbf{u}_F^+, \mathfrak{s})$$

and  $\text{CF}(\Sigma, \alpha, \beta, \mathbf{u}_F^+, \mathfrak{s})$  is generated by the elements  $\mathbf{u} \cdot \mathbf{x}$  where  $\mathbf{u} \in G(\mathbb{A}_F^+)$  is a monomial in  $\mathbb{A}_F^+$  and  $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$  is an intersection points such that

$$\mathfrak{s}(\mathbf{u} \cdot \mathbf{x}) := \mathfrak{s}(\mathbf{x}) + \chi(\mathbf{u}) = \mathfrak{s}.$$

For a relative class  $\mathfrak{s} \in \mathfrak{s}_0$ , let

$$\mathbf{u} = \prod_{i=1}^m u_i^{a_i} \prod_{j=1}^{\kappa} v_j^{b_j} \quad \text{and} \quad \mathbf{u}' = \prod_{i=1}^m u_i^{c_i} \prod_{j=1}^{\kappa} v_j^{d_j}$$

be non-zero monomials in  $G(\mathbb{A}_F^+)$  and  $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$  be intersection points such that  $\mathfrak{s}(\mathbf{u} \cdot \mathbf{x}) = \mathfrak{s}(\mathbf{u}' \cdot \mathbf{y}) = \mathfrak{s}$ . Then, we say  $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$  connects  $\mathbf{u} \cdot \mathbf{x}$  to  $\mathbf{u}' \cdot \mathbf{y}$  if

$$\begin{cases} a_i + \sum_{T_j \subset F_i} n_{z_j}(\phi) = c_i & 1 \leq i \leq m \\ b_j + n_{\bar{z}_j}(\phi) = d_j & 1 \leq j \leq \kappa. \end{cases}$$

Moreover, we define the relative grading of  $\mathbf{u} \cdot \mathbf{x}$  and  $\mathbf{u}' \cdot \mathbf{y}$  by

$$\text{gr}(\mathbf{u} \cdot \mathbf{x}, \mathbf{u}' \cdot \mathbf{y}) = \mu(\phi)$$

where  $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$  is a disk connecting  $\mathbf{u} \cdot \mathbf{x}$  to  $\mathbf{u}' \cdot \mathbf{y}$ .

**Lemma 6.3.** *The relative grading  $\text{gr}$  is well-defined and is independent of the choice of  $\phi$ . It induces a relative grading on the chain complex*

$$\text{CF}(M_F, T_F \sqcup \bar{T}_F, \mathbf{u}_F^+, \mathfrak{s})$$

*and the differential lowers this grading by one.*

**Proof.** Suppose that  $\phi, \phi' \in \pi_2(\mathbf{x}, \mathbf{y})$  connect  $\mathbf{u}.\mathbf{x}$  to  $\mathbf{u}'.\mathbf{y}$ . Then,

$$\mathcal{D} = \mathcal{D}(\phi) - \mathcal{D}(\phi') = \sum_{i=1}^n m_i \mathcal{P}_i + \sum_{k=1}^l n_k A_k$$

where  $\mathcal{P}_i$  is the periodic domain associated with  $H_i$  and  $\Sigma - \{\alpha\} = \sqcup_{k=1}^l A_k$ . From here we find

$$\begin{aligned} \mu(\phi) - \mu(\phi') &= \sum_{i=1}^n m_i \langle c_1(\mathfrak{s}_0), H_i \rangle + \sum_{k=1}^l n_k \langle c_1(\mathfrak{s}_0), H(A_k) \rangle \\ &= \sum_{k=1}^l n_k \langle c_1(\mathfrak{s}_0), H(A_k) \rangle. \end{aligned}$$

Moreover,  $\sum_{T_j \subset F_i} n_{z_j}(\mathcal{P}_k) = 0$  for every  $i$  and  $k$ . Hence,  $n_k = 0$  for all  $k$ , which implies that  $\mu(\phi) = \mu(\phi')$ .  $\square$

**Proposition 6.4.** *With respect to the relative grading defined above, the top-dimensional homology group in  $\mathrm{HF}(M_F, T_F \sqcup \bar{T}_F, \mathbf{u}_F^+, \mathfrak{s}_{F'})$  is isomorphic to  $\mathbb{A}_F^+$ . If  $\Theta_F$  denotes the generator of the top homology group, then for*

$$\begin{aligned} \phi : \mathrm{HF}(M_F, T_F \sqcup \bar{T}_F, \mathbf{u}_F^+, \mathfrak{s}_{F'}) &\rightarrow \mathrm{H}_*(\mathrm{CF}(M_F, T_F \sqcup \bar{T}_F, \mathbf{u}_F^+, \mathfrak{s}_{F'}) \otimes \mathbb{F}) \\ &= \mathrm{SFH}(X_F, \tau_F \sqcup \bar{\tau}_F, \mathfrak{s}_{F'}) = \mathbb{F} \end{aligned}$$

we have  $\phi(\Theta_F) = 1$ .

**Proof.** Let  $\pi : \{p_i\}_i \times [0, 1] \rightarrow [0, 1]$  denote the projection map. Consider a spanning set of arcs  $J = \{J_1, \dots, J_n\}$  on  $F$  such that the endpoints of each  $J_i$  lie on distinct components on  $T^+$  and if for some  $j < k$  an endpoint  $e_j$  of  $J_j$  and an endpoint  $e_k$  of  $J_k$  are on  $\{p_i\} \times [0, 1]$  then  $\pi(e_j) > \pi(e_k)$ . We call such a spanning set an *ordered* set of spanning arcs. If  $J$  is ordered, the result of surgery on  $(M^+, T^+ \sqcup \bar{T})$  along  $J_1, \dots, J_i$ , which is denoted by  $(M^i, T^i \sqcup \bar{T}^i)$ , is a tangle for  $i = 1, \dots, n$ .

For  $1 \leq i \leq n$ , let  $\mathfrak{s}_0^i$  denote the  $\mathrm{Spin}^c$  structure on  $M^i$  such that for any  $1 \leq j \leq i$  we have  $\langle c_1(\mathfrak{s}_0^i), H_j \rangle = 0$ . Thus,  $\mathfrak{s}_0^n = \mathfrak{s}_0$ . Assume that the connected components of  $T^i = \sqcup_{j=1}^i T_j^i$  are labeled such that  $T_j^i \cap \partial^+ M = \{p_j\}$ . For every  $1 \leq i \leq n$  we define a ring  $\mathbb{A}_i^+$  together with a map  $\mathbf{u}_i^+ : \pi_0(T^i \sqcup \bar{T}^i) \rightarrow \mathbb{A}_i^+$  inductively, such that  $[M^i, T^i \sqcup \bar{T}^i, \mathfrak{s}_0^i, \mathbf{u}_i^+]$  becomes an  $\mathbb{A}_i^+$ -tangle. For  $(M^0, T^0 \sqcup \bar{T}^0) = (M^+, T^+ \sqcup \bar{T})$ , let

$$\mathbb{A}_0^+ := \frac{\mathbb{Z}_2[\mathbf{v}'_1, \dots, \mathbf{v}'_\kappa, \mathbf{v}_1, \dots, \mathbf{v}_\kappa]}{\langle \mathbf{u}^k \mid g_k > 0 \rangle}$$

where  $\mathbf{u}^k = \prod_{p_j \in S_k} (\mathbf{v}_j \mathbf{v}'_j)$ . Then we define  $\mathbf{u}_0^+ : \pi_0(T^0 \sqcup \bar{T}^0) \rightarrow \mathbb{A}_0^+$  by

$$\mathbf{u}_0^+(T_j^0) = \mathbf{v}'_j \quad \text{and} \quad \mathbf{u}_0^+(\bar{T}_j^0) = \mathbf{v}_j \quad \text{for every } 1 \leq j \leq \kappa.$$

For every  $1 \leq i \leq n$  if  $J_i$  connects  $T_a^{i-1}$  and  $T_b^{i-1}$ , we define

$$\mathbb{A}_i^+ := \frac{\mathbb{A}_{i-1}^+}{\langle \mathbf{u}_{i-1}^+(T_a^{i-1}) - \mathbf{u}_{i-1}^+(T_b^{i-1}) \rangle}$$

and  $\mathbf{u}_i^+ : \pi_0(T^i \sqcup \bar{T}^i) \rightarrow \mathbb{A}_i^+$  is the map induced by  $\mathbf{u}_{i-1}^+$ . Note that  $\mathbb{A}_n^+ = \mathbb{A}_F^+$ .

We may construct an  $\mathbb{A}_i^+$ -diagram for  $\mathcal{T}^i = [M^i, T^i \sqcup \bar{T}^i, \mathbf{s}_0^i, \mathbf{u}_i^+]$  using an  $\mathbb{A}_{i-1}^+$ -diagram

$$H^{i-1} = (\Sigma^{i-1}, \boldsymbol{\alpha}^{i-1}, \boldsymbol{\beta}^{i-1}, \mathbf{u}_{i-1}^+ : \mathbf{z}^{i-1} \sqcup \bar{\mathbf{z}}^{i-1} \rightarrow \mathbb{A}_{i-1}^+, \mathbf{s}_0^{i-1})$$

for  $\mathcal{T}^{i-1}$  by setting

$$H^i = (\Sigma^i, \boldsymbol{\alpha}^i = \boldsymbol{\alpha}^{i-1} \cup \{\alpha_i\}, \boldsymbol{\beta}^i = \boldsymbol{\beta}^{i-1} \cup \{\beta_i\}, \mathbf{u}_i^+ : \mathbf{z}^i \sqcup \bar{\mathbf{z}}^i \rightarrow \mathbb{A}_i^+, \mathbf{s}_0^i),$$

where  $\Sigma^i$  is obtained from  $\Sigma^{i-1}$  by attaching a one-handle whose feet lie in a neighborhood of  $z_a$  and  $z_b$ , the marked points associated with  $T_a^{i-1}$  and  $T_b^{i-1}$ ,  $\alpha_i$  is the core of the one-handle and  $\beta_i$  is a Hamiltonian translate of  $\alpha_i$ . Moreover,  $\mathbf{z}^i = \mathbf{z}^{i-1}$ ,  $\bar{\mathbf{z}}^i = \bar{\mathbf{z}}^{i-1}$  and  $\mathbf{u}_i^+$  is the map induced by  $\mathbf{u}_{i-1}^+$ . Furthermore,  $\beta_i$  intersects  $\alpha_i$  in a pair of canceling intersection points and the area bounded between  $\alpha_i$  and  $\beta_i$  gives the 2-chain  $\mathcal{P} = D_i^+ - D_i^-$  with  $\partial \mathcal{P} = \alpha_i - \beta_i$ , where the connected components  $D_i^+$  and  $D_i^-$  of  $\Sigma^i - \boldsymbol{\alpha}^i - \boldsymbol{\beta}^i$  are bigons such that one of them intersects  $\mathbf{z}$  in  $z_a$  while the other one intersects  $\mathbf{z}$  in  $z_b$ .

Therefore, starting from the  $\mathbb{A}_0^+$ -diagram

$$H^0 = (\Sigma^0 = S, \boldsymbol{\alpha}^0 = \emptyset, \boldsymbol{\beta}^0 = \emptyset, \mathbf{u}_0^+ : \mathbf{z} \sqcup \bar{\mathbf{z}} \rightarrow \mathbb{A}_0^+, \mathbf{s}_0^0)$$

with  $\mathbf{z} = \partial^+ T$  for the product  $\mathbb{A}_0^+$ -tangle  $\mathcal{T}^0$ , we obtain an  $\mathbb{A}_F^+$ -diagram

$$H := H^n = (\Sigma, \boldsymbol{\alpha} = \{\alpha_1, \dots, \alpha_n\}, \boldsymbol{\beta} = \{\beta_1, \dots, \beta_n\}, \mathbf{u}_F^+ : \mathbf{z} \sqcup \bar{\mathbf{z}} \rightarrow \mathbb{A}_F^+, \mathbf{s}_0)$$

for  $\mathcal{T}^+$ . Any pair of curves  $(\alpha_i, \beta_i)$  intersect in a pair of points  $x_i^+$  and  $x_i^-$ , so that the bi-gons  $D_i^+$  and  $D_i^-$  connect  $x_i^+$  to  $x_i^-$ . Thus, corresponding to any map  $\epsilon : \{1, \dots, n\} \rightarrow \{+, -\}$  we have an intersection point

$$\mathbf{x}^\epsilon = \{x_1^{\epsilon(1)}, \dots, x_n^{\epsilon(n)}\} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta.$$

By an argument similar to the argument of Proposition 5.2, the map

$$\mathfrak{f}_i : \text{CF}(\Sigma^{i-1}, \boldsymbol{\alpha}^{i-1}, \boldsymbol{\beta}^{i-1}, \mathbf{u}_{i-1}^+, \mathbf{s}_0^{i-1}) \otimes \mathbb{A}_i^+ \rightarrow \text{CF}(\Sigma^i, \boldsymbol{\alpha}^i, \boldsymbol{\beta}^i, \mathbf{u}_i^+, \mathbf{s}_0^i)$$

which is defined by  $\mathfrak{f}_i(\mathbf{y}) = \mathbf{y} \times \{x_i^+\}$ , is a chain map if the path of almost complex structures is chosen correctly. In particular, the intersection point  $\Theta_F^+ := \{x_1^+, \dots, x_n^+\}$  is closed in the chain complex

$$\text{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{u}_F^+, \mathbf{s}_0) = \text{CF}(\mathcal{T}^+),$$

i.e. we have  $\partial \Theta_F^+ = 0$ .



For every  $1 \leq i \leq n$ , let  $(F'_i, \partial F'_i) \subset (M^i, \partial M^i \cup T^i \cup \bar{T}^i)$  denote the embedded, oriented surface obtained from  $D$  by attaching the embedded one handles  $D'_1, \dots, D'_i$ . Each  $F'_i$  is then a decomposing surface in  $(M^i, T^i \sqcup \bar{T}^i)$ . We may inductively construct a closed subsurface  $P^i \subset \Sigma^i$  such that the Heegaard diagram  $H^i$ , together with  $P^i$ , is a diagram adapted to  $F'_i$  in the sense of [Ju2, Definition 4.3]. More precisely,  $P^i$  is a closed subsurface of  $\Sigma^i$  such that the boundary of  $P^i$  is a union of polygons, whose vertices are  $P^i \cap (\mathbf{z} \sqcup \bar{\mathbf{z}})$  and its edges are decomposed as  $\partial P^i = A^i \cup B^i$  where

$$A^i \cap B^i \subset \mathbf{z} \sqcup \bar{\mathbf{z}}, \quad \alpha^i \cap B^i = \emptyset \quad \text{and} \quad \beta^i \cap A^i = \emptyset.$$

Finally, the equivalence class of  $F'_i$  is given by smoothing the corners of

$$(P^i \times \{1/2\}) \cup (A^i \times [1/2, 1]) \cup (B^i \times [0, 1/2]) \subset (M^i, T^i \sqcup \bar{T}^i).$$

For every  $i$ , suppose that  $J_i$  connects  $T_{a_i}^{i-1}$  to  $T_{b_i}^{i-1}$ . For  $i = 1$ ,  $F'_1$  is obtained from  $D$  by attaching a one-handle to  $T_{a_1}^0$  and  $T_{b_1}^0$ . Thus,  $P^1$  is a union of a rectangle whose vertices are  $z_{a_1}, z_{b_1}, \bar{z}_{a_1}$  and  $\bar{z}_{b_1}$  and contains the intersection point  $x_1^-$  together with  $n - 2$  bigons disjoint from  $\alpha^1$  and  $\beta^1$  curves, whose vertices are  $z_j$  and  $\bar{z}_j$  for  $j \neq a_1, b_1$ . For any  $i > 1$ , the subsurface  $P^i$  may be constructed from  $P^{i-1}$  by attaching an embedded one-handle in  $\Sigma^i$  which intersects  $\alpha_i$  and  $\beta_i$  and contains  $x_i^-$  as illustrated in Figure 5.

Thus, for  $i = n$ ,  $\Theta_F^+$  is the only intersection point that does not intersect  $P^n$ . As a result,  $\Theta_F^+$  is the only intersection point for which  $\underline{\mathbf{s}}(\Theta_F^+) = \underline{\mathbf{s}}_{F'}$  and thus  $\Theta_F^+$  is the generator of

$$\text{SFH}(M_F, T_F \sqcup \bar{T}_F, \underline{\mathbf{s}}_{F'}) = \mathbb{F}.$$

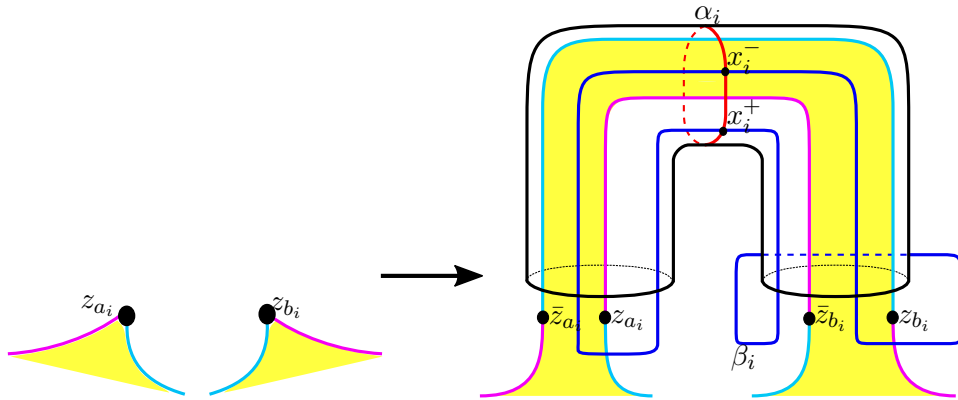


FIGURE 5. The  $i$ -th one-handle is attached over the marked points  $z_{a_i}$  and  $z_{b_i}$ . The curves  $\alpha_i$  and  $\beta_i$ , their intersection points  $x_i^+$  and  $x_i^-$  and the modification changing  $P^{i-1}$  to  $P^i$  are illustrated.

For every  $\epsilon : \{1, \dots, n\} \rightarrow \{+, -\}$ , there is a positive disk

$$\phi \in \pi_2(\Theta_F^+, \mathbf{x}^\epsilon) \quad \text{with} \quad \mu(\phi) = \#\{i \mid \epsilon(i) = -\}.$$

Furthermore,  $\underline{s}(\mathbf{u}_F^+(\phi). \mathbf{x}^\epsilon) = \underline{s}_{F'}$  and if for a monomial  $\mathbf{u} \in \mathbb{A}_F^+$  we have  $\underline{s}(\mathbf{u}. \mathbf{x}^\epsilon) = \underline{s}_{F'}$  then  $\mathbf{u} = \mathbf{u}_F^+(\phi). \tilde{\mathbf{u}}$  for some  $\tilde{\mathbf{u}} \in \mathbb{A}_F^+$ . Hence,  $\Theta_F^+$  generates the top-dimensional homology group in

$$\text{HF}(M_F, T_F \sqcup \overline{T}_F, \mathbf{u}_F^+, \underline{s}_{F'})$$

with respect to the relative grading defined above. The above observations complete the proof of the proposition.  $\square$

**Definition 6.5.** *The unique element  $\Theta_F$  of*

$$\text{HF}_{\mathbb{A}}(\mathcal{T}_F) = H_{\star}(\text{CF}(\mathcal{T}^+) \otimes \mathbb{A}).$$

*which is determined by the top generator  $\Theta_F^+$  in the right-hand-side homology group, is called the distinguished generator of  $\text{HF}_{\mathbb{A}}(\mathcal{T}_F)$ .*

**6.3. Framed arcs, framed knots and the cobordism map.** Let  $\mathcal{W} = (W, F) = \mathcal{W}(\mathbb{I}, \mathbb{S})$  be a stable cobordism from the balanced tangle  $(M, T)$  to another balanced tangle  $(M', T')$ , which corresponds to an acceptable set  $\mathbb{I} = \{\mathbb{I}_1, \dots, \mathbb{I}_n\}$  of framed arcs in  $(M, T)$  and a set

$$\mathbb{S} = \{\mathbb{S}_1, \dots, \mathbb{S}_m\}$$

of embedded framed 1-spheres in the complement of  $T$  and the framed arcs  $\mathbb{I}$  in  $M$ . Each  $\mathbb{S}_i$  is determined by a framing on a knot  $K_i$  and each  $\mathbb{I}_i$  is determined by a framing on an arc  $I_i$ . Moreover,  $(M', T') = (M(\mathbb{S}), T(\mathbb{I}))$  where  $T(\mathbb{I})$  is constructed from band surgery on  $T$  along  $\mathbb{I}$  and  $M(\mathbb{S})$  is constructed from  $M$  by surgery along the framed link  $\mathbb{S}$ .

**Definition 6.6.** *Given a balanced tangle  $(M, T)$  and a set  $\mathbb{I}$  of framed arcs for  $\mathcal{T}$ , a stalk  $s(\mathbb{S}) = \{s_1, \dots, s_m\}$  for the framed link  $\mathbb{S} = \{\mathbb{S}_1, \dots, \mathbb{S}_m\}$  (disjoint from  $\mathbb{I}$ ) in  $\mathcal{T}$  is a set of embedded arcs in  $M - T - \mathbb{I}$ , such that  $s_i$  connects the knot  $K_i = \mathbb{S}_i(S^1 \times \{0\})$  to  $\partial^+ M - T$ .*

For every  $i$ , if  $T_i$  intersects  $\mathbb{I}$ , let  $r_i \subset T_i$  be the segment where

$$\partial^+ r_i \subset \partial^+ M, \quad \partial^- r_i \subset T_i \cap \mathbb{I} \quad \text{and} \quad (T_i - r_i) \cap \mathbb{I} = \emptyset.$$

Let  $r_i = \emptyset$  if  $T_i \cap \mathbb{I} = \emptyset$ . Set

$$B = B(\mathbb{I}) \cup B(\mathbb{S}, s(\mathbb{S})) = \left( \prod_{i=1}^n I_i \right) \cup \left( \prod_{i=1}^{\kappa} r_i \right) \cup \left( \prod_{i=1}^m (K_i \cup s_i) \right).$$

Consider a small tubular neighborhood  $\text{nd}(B)$  of  $B$  and let  $A_{\mathbb{I}} = \prod_{i=1}^n A_i$  where  $A_i$  is a sphere with 4 boundary components which is the intersection of an enlarged neighborhood of  $I_i$  with  $\partial(\text{nd}(B))$ . Such neighborhoods are

illustrated in Figure 6. The intersection of  $T$  with  $M - \text{nd}(B)$  defines a new tangle

$$(M'', T'') = (M - \text{nd}(B), T \cap M'').$$

**Definition 6.7.** *A Heegaard triple subordinate to the framed arcs  $\mathbb{I}$ , the framed link  $\mathbb{S}$ , and the stalk  $s(\mathbb{S})$  for the balanced tangle  $(M, T)$  is a Heegaard triple*

$$(\Sigma, \alpha = \{\alpha_1, \dots, \alpha_\ell\}, \beta = \{\beta_1, \dots, \beta_\ell\}, \gamma = \{\gamma_1, \dots, \gamma_\ell\}, \mathbf{z})$$

*satisfying the following conditions:*

- (1)  $(\Sigma, \alpha, \{\beta_{n+m+1}, \dots, \beta_\ell\}, \mathbf{z})$  is a Heegaard diagram for  $(M'', T'')$ .
- (2) For  $i = m+n+1, \dots, \ell$ ,  $\gamma_i$  is obtained by a small Hamiltonian isotopy from  $\beta_i$  supported away from the marked points, so that  $|\beta_i \cap \gamma_i| = 2$ .
- (3) For  $i = 1, \dots, m$  the curves  $\beta_{n+i}$  and  $\gamma_{n+i}$  lie on the punctured torus  $\partial \mathbb{S}_i - (\mathbb{S}_i \cap \text{nd}(s(\mathbb{S})))$ . The curve  $\beta_{n+i}$  represents the meridian of  $K_i$  while  $\gamma_{n+i}$  represents the framing, and they meet in a single transverse intersection point.
- (4) The curves  $\beta_j$  and  $\gamma_j$  on  $\partial^+ M''$  lie the punctured sphere  $A_j$  for  $j = 1, \dots, n$ . The curve  $\beta_j$  represents the meridian of  $\mathbb{I}_j$  and meets  $\gamma_j$  in two transverse intersection points, while  $\gamma_j$  is obtained from  $\beta_j$  by an isotopy corresponding to the framing of  $\mathbb{I}_j$  (which crosses two of the boundary components of  $A_j$ ), as illustrated in Figure 6.
- (5) The Heegaard diagrams  $(\Sigma, \alpha, \beta, \mathbf{z})$  and  $(\Sigma, \alpha, \gamma, \mathbf{z})$  are diagrams for  $(M, T)$  and  $(M', T')$ , respectively.

We say that a Heegaard triple is subordinate to the framed arcs  $\mathbb{I}$  and the framed link  $\mathbb{S}$  if it is subordinate to the framed arcs  $\mathbb{I}$  and the framed link  $\mathbb{S}$  and some stack  $s(\mathbb{S})$  for  $\mathbb{S}$ .

The existence of Heegaard triples subordinate to an acceptable set of framed arcs  $\mathbb{I}$  and a framed link  $\mathbb{S}$  for a stable cobordism  $\mathcal{W}$  as above and the correspondence between different such Heegaard triples is addressed in the following lemma.

**Lemma 6.8.** *Let  $\mathbb{I} = \{\mathbb{I}_1, \dots, \mathbb{I}_n\}$  denote an acceptable set of framed arcs in the balanced tangle  $(M, T)$  and  $\mathbb{S} = \{\mathbb{S}_1, \dots, \mathbb{S}_m\}$  denote a framed link in  $M - T - \mathbb{I}$ . Then there is a Heegaard triple subordinate to  $\mathbb{I}$  and  $\mathbb{S}$ . Every two such triples may be connected (after composing with a diffeomorphism of the diagram) by a sequence of following moves, all supported away from the set  $\mathbf{z}$  of marked points:*

- (1) *Isotopies and handle slides among  $\{\alpha_1, \dots, \alpha_\ell\}$*
- (2) *Isotopies and handle slides among  $\{\beta_{n+m+1}, \dots, \beta_\ell\}$  while carrying the corresponding isotopy or handle slide along  $\{\gamma_{n+m+1}, \dots, \gamma_\ell\}$ .*
- (3) *Stabilization (and destabilization); i.e. taking the connected sum of the Heegaard triple with a triple  $(E, \alpha, \beta, \gamma)$ , where  $E$  is a surface of genus one,  $|\alpha \cap \beta| = 1$ , and  $\gamma$  is obtained by a small Hamiltonian isotopy from  $\beta$  such that  $|\beta \cap \gamma| = 2$ .*

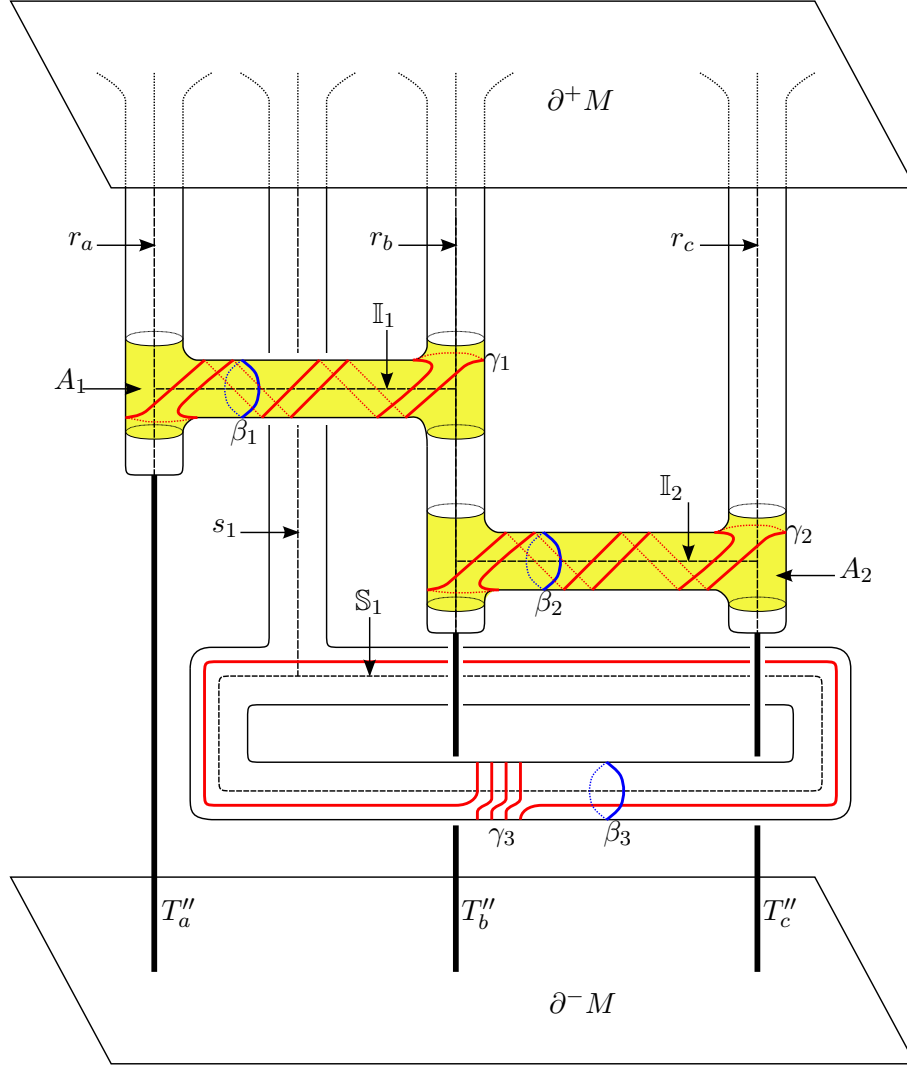


FIGURE 6. A Heegaard diagram subordinate to a framed knot  $S_1$  and a pair of framed arcs  $I_1$  and  $I_2$  with one end point on the same strand  $T_b$  of  $T$ , and the other ends on the strands  $T_a$  and  $T_c$ . A tubular neighborhood of the union of  $K_1, s_1, I_1, I_2, r_a, r_b$  and  $r_c$  is deleted to obtain the tangle  $(M'', T'')$ . Attaching disks to the meridians  $\beta_1$  and  $\beta_2$  of  $I_1$  and  $I_2$  and the meridian  $\beta_3$  of  $S_1$  gives a Heegaard diagram for  $(M, T)$  while the framings of  $I_1, I_2$  and  $S_1$  determine the curves  $\gamma_1, \gamma_2$  and  $\gamma_3$ . The curves  $\gamma_1$  and  $\gamma_2$  live in the 4-punctured spheres  $A_1$  and  $A_2$ , respectively.

- (4) *Isotopies or handle slides of  $\beta_j$  along the curves in  $\{\beta_{n+m+1}, \dots, \beta_\ell\}$  for  $j = 1, \dots, n + m$ .*

- (5) *Isotopy or handle slides of  $\gamma_j$  along the curves in  $\{\gamma_{n+m+1}, \dots, \gamma_\ell\}$  for  $j = 1, \dots, n+m$ .*
- (6) *Handle-slide of a curve in  $\{\beta_{n+m+1}, \dots, \beta_\ell\}$  along some  $\beta_{n+j}$  for  $j = 1, \dots, m$ , while doing a handle-slide of the corresponding curve in  $\{\gamma_{n+m+1}, \dots, \gamma_\ell\}$  along  $\gamma_{n+j}$ .*

**Proof.** Given the stack  $s(\mathbb{S})$  for  $\mathbb{S}$ , the proof of Lemma 4.5 from [OS4] may be used to show that Heegaard diagrams subordinate to  $\mathbb{I}$ ,  $\mathbb{S}$  and  $s(\mathbb{S})$  exist and that for every pair  $H, H'$  of such diagrams  $H$  may be changed to  $H'$  via a sequence of moves of types 1, 2, 3, 4 and 5 in the statement of the lemma. If the stacks  $s(\mathbb{S})$  and  $s'(\mathbb{S})$  are different, the proof of Lemma 4.8 from [OS4] implies that there is a Heegaard diagram  $H$  as above subordinate to  $\mathbb{I}$ ,  $\mathbb{S}$  and  $s(\mathbb{S})$  and a Heegaard diagram  $H'$  subordinate to  $\mathbb{I}$ ,  $\mathbb{S}$  and the stalk  $s'(\mathbb{S})$  such that the following is true. There is a sequence of handle slides of some particular curves in  $\{\beta_{n+m+1}, \dots, \beta_\ell\}$  over the curves  $\beta_{n+j}$  for  $j = 1, \dots, m$  and other curves in  $\{\beta_{n+m+1}, \dots, \beta_\ell\}$  (and a corresponding sequence of handle slides for  $\gamma$ ) which change  $H$  to  $H'$ . The lemma then follows.  $\square$

Let  $H = (\Sigma, \alpha, \beta, \gamma, \mathbf{z})$  be a Heegaard triple subordinated to the framed arc  $\mathbb{I}$  and the framed link  $\mathbb{S}$  as above. Associated with  $H$  we have a cobordism

$$\mathcal{W}_H = (W_{\alpha\beta\gamma}, F_{\mathbf{z}}) : (M, T) \sqcup (M_{\beta\gamma}, T_{\beta\gamma}) \rightsquigarrow (M(\mathbb{S}), T(\mathbb{I})),$$

where  $(M_{\beta\gamma}, T_{\beta\gamma})$  is the balanced tangle determined by the Heegaard diagram  $(\Sigma, \beta, \gamma, \mathbf{z})$ .

Let  $(M^+, T^+) := \partial^+ \mathcal{W}(\mathbb{I}, \mathbb{S})$  and  $(M_F, T_F)$  denote the associated balanced tangles as in Subsection 6.1. The tangle  $(M_{\beta\gamma}, T_{\beta\gamma})$  is constructed from  $(M_F, T_F)$  by attaching  $\ell - n - m$  one-handles to  $M_F - T_F$ . Let the spheres  $S_1, \dots, S_{\ell-n-m}$  denote the cores of these one-handles.

Associated with every framed arc  $\mathbb{I}_i$  a properly embedded arc  $J_i \subset F$  is obtained as follows. The framed arc  $\mathbb{I}_i$  determines an embedded arc  $J_i$  on  $F$  with endpoints on  $T$ . These endpoints may first be moved to  $\partial^+ T$  and then to the interior of  $T^+$  by an isotopy. More precisely, after applying a smooth isotopy supported in a neighborhood of  $\partial F$  on the arcs  $J = \{J_1, \dots, J_n\}$  we may assume that

$$\coprod_{i=1}^n \partial J_i \subset F \cap M^+ = T^+.$$

We call  $\mathbb{I} = (\mathbb{I}_1, \dots, \mathbb{I}_n)$  an *ordered* set of framed arcs if the corresponding arcs  $(J_1, \dots, J_n)$  are ordered. It is relatively easy to show that every acceptable set of framed arcs  $\mathbb{I}$  may be changed to an ordered set of framed arcs by performing a number of arc slides.

Let  $\text{nd}(J) = \sqcup_{i=1}^n \text{nd}(J_i)$  be a union of disjoint small tubular neighborhoods around  $J_1, \dots, J_n$ . Then,

$$(W - \text{nd}(J), F \cap (W - \text{nd}(J)))$$

gives a cobordism from  $(M, T) \sqcup (M_F, T_F)$  to  $(M(\mathbb{S}), T(\mathbb{I}))$ .

**Lemma 6.9.** *Under the above assumptions, after attaching 3-handles along the spheres  $S_1, \dots, S_{\ell-n-m} \subset M_{\beta\gamma}$  to  $\mathcal{W}_H$ , the cobordism*

$$(W - \text{nd}(J), F \cap (W - \text{nd}(J)))$$

*is obtained.*

**Proof.** Denote the compression body obtained by attaching disks to  $\beta$  by  $C(\beta)$ . Without loss of generality, we may assume that

$$\mathbb{I}_i, \mathbb{S}_j \subset C(\beta) \quad \text{for } i \in \{1, \dots, n\}, j \in \{1, \dots, m\}.$$

Furthermore, we may consider an identification

$$W = (M \times [0, 1]) \cup_{\{1\} \times \mathbb{S}} \left( \bigcup_{i=1}^m D^2 \times D^2 \right)$$

such that the restriction map  $\pi_2|_F$  induced by projection of  $M \times [0, 1]$  over the second factor, is a Morse function which only has critical points of index one. Moreover, we may assume that

$$\text{Crit}(\pi_2|_F) \subset M \times \left\{ \frac{1}{2} \right\}.$$

Thus,  $T_{\frac{1}{2}} := F \cap (M \times \{\frac{1}{2}\})$  is a properly embedded, oriented singular 1-dimensional submanifold of  $M \times \{\frac{1}{2}\}$ . Associated with every framed arc  $\mathbb{I}_i$  we have a singular point

$$q_i \subset T_{\frac{1}{2}} \cap \left( C(\beta) \times \left\{ \frac{1}{2} \right\} \right) \subset F.$$

Let  $B'(\mathbb{I}) \subset T_{\frac{1}{2}}$  be the subspace corresponding to  $B(\mathbb{I})$ . Let  $N', N_{1/2} \subset W$  be small open neighborhoods around  $B'(\mathbb{I})$  and  $C(\beta) \times \{\frac{1}{2}\}$ , respectively. Then

$$\begin{aligned} (W - N', F \cap (W - N')) &= (W - \text{nd}(J), F \cap (W - \text{nd}(J))) \quad \text{and} \\ (W - N_{1/2}, F \cap (W - N_{1/2})) &= \mathcal{W}_H. \end{aligned}$$

Thus,  $(W - \text{nd}(J), F \cap (W - \text{nd}(J)))$  is obtained from  $\mathcal{W}_H$  by attaching 3-handles along  $S_1, \dots, S_{\ell-n-m} \subset M_{\beta\gamma}$ .  $\square$

In the above situation, Lemma 6.9 implies that associated with every Heegaard triple  $H$  subordinate to the family of framed arcs  $\mathbb{I}$  and the framed link  $\mathbb{S}$  we have a restriction map

$$r : \text{Spin}^c(\mathcal{W}(\mathbb{I}, \mathbb{S})) \rightarrow \text{Spin}^c(\mathcal{W}_H).$$

Abusing the notation, let  $\mathfrak{s}_0 \in \text{Spin}^c(M_{\beta\gamma})$  denote the  $\text{Spin}^c$  class obtained from  $\mathfrak{s}_0 \in \text{Spin}^c(M_F)$  which satisfies

$$\langle c_1(\mathfrak{s}_0), S_i \rangle = 0 \quad \forall \quad i \in \{1, 2, \dots, \ell - n - m\}.$$

**Lemma 6.10.** *For every  $\mathfrak{t} \in \text{Spin}^c(\mathcal{W}(\mathbb{I}, \mathbb{S}))$ , the restriction  $r(\mathfrak{t})|_{M_{\beta\gamma}}$  is the  $\text{Spin}^c$  class  $\mathfrak{s}_0 \in \text{Spin}^c(M_{\beta\gamma})$ .*

**Proof.** This is straightforward.  $\square$

Given a  $\text{Spin}^c$  class  $\mathfrak{t} \in \text{Spin}^c(\mathcal{W}(\mathbb{I}, \mathbb{S}))$  we abuse the notation and call the Heegaard triple  $(\Sigma, \alpha, \beta, \gamma, \mathfrak{u})$   $\mathfrak{t}$ -admissible if it is  $r(\mathfrak{t})$ -admissible. Note that every diagram  $(\Sigma, \alpha, \beta, \gamma, \mathfrak{u})$  subordinate to  $\mathbb{I}$  and  $\mathbb{S}$  may be transformed to a  $\mathfrak{t}$ -admissible Heegaard diagram by performing isotopies on the curves in  $\alpha$ , following a procedure similar to [AE, Section 4.2]. Let us assume that the Heegaard diagram  $(\Sigma, \alpha, \beta, \gamma, \mathfrak{u})$  subordinate to  $\mathbb{I}$  and  $\mathbb{S}$  is  $\mathfrak{t}$ -admissible. We then call  $H = (\Sigma, \alpha, \beta, \gamma, \mathfrak{u}, \mathfrak{t})$  an  $\mathbb{A}$ -diagram for

$$\mathcal{C} = [W, F, \mathfrak{t}, \mathfrak{u}_F] : \mathcal{T} = [M, T, \mathfrak{s}, \mathfrak{u}] \rightsquigarrow \mathcal{T}' = [M', T', \mathfrak{s}', \mathfrak{u}']$$

subordinate to the framed arcs  $\mathbb{I}$  and the framed link  $\mathbb{S}$ . The  $\mathbb{A}$ -diagram  $H$  determines a holomorphic triangle map

$$\mathfrak{f}_H : \text{CF}(\Sigma, \alpha, \beta, \mathfrak{u}, \mathfrak{s}) \otimes_{\mathbb{A}} \text{CF}(\Sigma, \beta, \gamma, \mathfrak{u}, \mathfrak{s}_0) \longrightarrow \text{CF}(\Sigma, \alpha, \gamma, \mathfrak{u}, \mathfrak{s}').$$

From the Heegaard diagram  $(\Sigma, \beta, \gamma, \mathfrak{z})$  for  $(M_{\beta\gamma}, T_{\beta\gamma})$  we may construct a Heegaard diagram  $H' = (\Sigma', \beta', \gamma', \mathfrak{z})$  for  $(M_F, T_F)$  where  $\Sigma'$  is obtained by surgering out  $\beta_{n+m+1}, \dots, \beta_\ell$ , while

$$\beta' = \{\beta_1, \dots, \beta_{n+m}\} \quad \text{and} \quad \gamma' = \{\gamma_1, \dots, \gamma_{n+m}\}.$$

Similarly, we may construct another Heegaard diagram  $H'' = (\Sigma'', \beta'', \gamma'', \mathfrak{z})$  for  $(M_F, T_F)$  where  $\Sigma''$  is obtained from  $\Sigma'$  by surgering out  $\beta_{n+1}, \dots, \beta_{n+m}$ , while

$$\beta'' = \{\beta_1, \dots, \beta_n\} \quad \text{and} \quad \gamma'' = \{\gamma_1, \dots, \gamma_n\}.$$

Note that  $H'$  is obtained from  $H''$  by  $m$  stabilizations. Let us denote the map associated with the aforementioned stabilization and then attaching one-handles by

$$\mathfrak{f}_{\beta\gamma} : \text{HF}(\Sigma'', \beta'', \gamma'', \mathfrak{u}, \mathfrak{s}_0) \rightarrow \text{HF}(\Sigma, \beta, \gamma, \mathfrak{u}, \mathfrak{s}_0)$$

Then, we set  $\Theta_{\beta\gamma} := \mathfrak{f}_{\beta\gamma}(\Theta_F)$  and define

$$\begin{aligned} \mathfrak{f}_{\mathbb{I}, \mathbb{S}, \mathfrak{t}} &= \mathfrak{f}_{\mathbb{I}, \mathbb{S}, H} : \text{HF}(\Sigma, \alpha, \beta, \mathfrak{u}, \mathfrak{s}) \longrightarrow \text{HF}(\Sigma, \alpha, \gamma, \mathfrak{u}, \mathfrak{s}') \\ \mathfrak{f}_{\mathbb{I}, \mathbb{S}, \mathfrak{t}}(\mathbf{x}) &:= \mathfrak{f}_H(\mathbf{x} \otimes \Theta_{\beta\gamma}) \quad \forall \quad \mathbf{x} \in \text{HF}(\Sigma, \alpha, \beta, \mathfrak{u}, \mathfrak{s}). \end{aligned}$$

**6.4. Independence from the choice of Heegaard diagrams.** Let us assume that

$$H = (\Sigma, \alpha, \beta, \gamma, \mathbf{u}, \mathbf{t}) \quad \text{and} \quad H' = (\Sigma', \alpha', \beta', \gamma', \mathbf{u}', \mathbf{t})$$

are  $\mathbb{A}$ -diagrams for  $\mathcal{C}$  subordinate to the acceptable set of framed arcs  $\mathbb{I}$  and the framed link  $\mathbb{S}$ . Lemma 6.8 implies that there is a sequence of Heegaard moves

$$H = H_0 \xrightarrow{\mathfrak{h}_1} H_1 \xrightarrow{\mathfrak{h}_2} \dots \xrightarrow{\mathfrak{h}_r} H_r = H'$$

of the type specified in the aforementioned lemma which change one diagram to the other. Let us assume that  $H_i = (\Sigma_i, \alpha_i, \beta_i, \gamma_i, \mathbf{u}_i, \mathbf{t})$  are  $\mathbb{A}$ -diagrams. Associated with each Heegaard move  $\mathfrak{h}_i$ , we obtain the chain homotopy equivalences

$$\begin{aligned} \Phi_{\mathfrak{h}_i} &: \text{CF}(\Sigma_{i-1}, \alpha_{i-1}, \beta_{i-1}, \mathbf{u}_{i-1}, \mathfrak{s}) \rightarrow \text{CF}(\Sigma_i, \alpha_i, \beta_i, \mathbf{u}_i, \mathfrak{s}), \\ \Psi_{\mathfrak{h}_i} &: \text{CF}(\Sigma_{i-1}, \alpha_{i-1}, \gamma_{i-1}, \mathbf{u}_{i-1}, \mathfrak{s}') \rightarrow \text{CF}(\Sigma_i, \alpha_i, \gamma_i, \mathbf{u}_i, \mathfrak{s}'). \end{aligned}$$

If  $\underline{\mathfrak{h}} = (\mathfrak{h}_1, \dots, \mathfrak{h}_r)$  denotes the sequence of moves relating  $H$  to  $H'$ , define

$$\Phi_{\underline{\mathfrak{h}}} = \Phi_{\mathfrak{h}_r} \circ \dots \circ \Phi_{\mathfrak{h}_1} \quad \text{and} \quad \Psi_{\underline{\mathfrak{h}}} = \Psi_{\mathfrak{h}_r} \circ \dots \circ \Psi_{\mathfrak{h}_1}.$$

**Theorem 6.11.** *Let  $\mathcal{C} = [W, F, \mathbf{t}, \mathbf{u}_F]$  be an  $\mathbb{A}$ -cobordism defined by an acceptable set  $\mathbb{I}$  of framed arcs and a framed link  $\mathbb{S}$  for balanced tangle  $(M, T)$  and  $\mathbf{t} \in \text{Spin}^c(W)$ . Suppose that*

$$H = (\Sigma, \alpha, \beta, \gamma, \mathbf{u}, \mathbf{t}) \quad \text{and} \quad H' = (\Sigma', \alpha', \beta', \gamma', \mathbf{u}', \mathbf{t})$$

are  $\mathbb{A}$ -diagrams for  $\mathcal{C}$  subordinate to  $\mathbb{I}$  and  $\mathbb{S}$  as above which are related by the sequence  $\underline{\mathfrak{h}}$  of Heegaard moves. Then the following diagram is commutative:

$$\begin{array}{ccc} \text{HF}(\Sigma, \alpha, \beta, \mathbf{u}, \mathfrak{s}) & \xrightarrow{\mathfrak{f}_{\mathbb{I}, \mathbb{S}, H}} & \text{HF}(\Sigma, \alpha, \gamma, \mathbf{u}, \mathfrak{s}') \\ \Phi_{\underline{\mathfrak{h}}} \downarrow & & \downarrow \Psi_{\underline{\mathfrak{h}}} \\ \text{HF}(\Sigma', \alpha', \beta', \mathbf{u}', \mathfrak{s}) & \xrightarrow{\mathfrak{f}_{\mathbb{I}, \mathbb{S}, H'}} & \text{HF}(\Sigma', \alpha', \gamma', \mathbf{u}', \mathfrak{s}') \end{array}$$

**Proof.** We only need to consider the case where  $H'$  is obtained by a single move  $\mathfrak{h}$  from  $H$ . Let us first consider the square corresponding to a Heegaard move  $\mathfrak{h}$  which changes  $\alpha$  to  $\alpha'$  and keeps  $\Sigma, \beta, \gamma$  and  $\mathbf{z}$  unchanged. The  $\text{Spin}^c$  structure  $\mathbf{t}$  determines a coherent system of  $\text{Spin}^c$  structures on the Heegaard 4-tuple  $(\Sigma, \alpha', \alpha, \beta, \gamma, \mathbf{z})$ , which will still be denoted by  $\mathbf{t}$ , abusing the notation. There is a holomorphic square map corresponding to the  $\mathbb{A}$ -diagram  $(\Sigma, \alpha', \alpha, \beta, \gamma, \mathbf{u}, \mathbf{t})$  which will be denoted by  $\mathfrak{S}_{\mathfrak{h}}$ . Considering different possible degenerations of a square either to a bigon and a square or to a pair of triangles gives the relation

$$\Psi_{\mathfrak{h}} \circ \mathfrak{f}_{\mathbb{I}, \mathbb{S}, H} - \mathfrak{f}_{\mathbb{I}, \mathbb{S}, H'} \circ \Phi_{\mathfrak{h}} = \mathfrak{S}_{\mathfrak{h}} \circ d + d \circ \mathfrak{S}_{\mathfrak{h}},$$



following the standard arguments in Heegaard Floer theory.

On the other hand, if  $\mathfrak{h}$  changes both  $\beta$  and  $\gamma$  simultaneously (to  $\beta'$  and  $\gamma'$ ) we obtain a pair of holomorphic square maps  $\mathfrak{S}_i = \mathfrak{S}_{\mathfrak{h},i}$ ,  $i = 1, 2$  which correspond to the Heegaard quadruples

$$(\Sigma, \alpha, \beta, \beta', \gamma', u, t) \text{ and } (\Sigma, \alpha, \beta, \gamma, \gamma', u, t),$$

respectively. Denote the maps corresponding to the Heegaard subdiagrams

$$(\Sigma, \alpha, \beta, \gamma', u, t), (\Sigma, \beta, \gamma, \gamma', u, t) \text{ and } (\Sigma, \beta, \beta', \gamma', u, t)$$

by  $\mathfrak{f}_{\alpha, \beta, \gamma'}$ ,  $\mathfrak{f}_{\beta, \gamma, \gamma'}$  and  $\mathfrak{f}_{\beta, \beta', \gamma'}$ , respectively.

The images of the distinguished generator  $\Theta_{\beta\gamma} \otimes \Theta_{\gamma\gamma'}$  under  $\mathfrak{f}_{\beta, \gamma, \gamma'}$  and the distinguished generator  $\Theta_{\beta\beta'} \otimes \Theta_{\beta'\gamma'}$  under  $\mathfrak{f}_{\beta, \beta', \gamma'}$  is the distinguished generator  $\Theta_{\beta\gamma'}$  which corresponds to  $(\Sigma, \beta, \gamma', u, t)$ . This should be done independently for each one of the Heegaard moves. The proofs follow from the standard arguments in Heegaard Floer theory since the Heegaard triples  $(\Sigma, \beta, \gamma, \gamma', \mathbf{z})$  and  $(\Sigma, \beta, \beta', \gamma')$  have standard forms.

Let us abuse the notation and denote  $\mathfrak{f}_{\alpha, \beta, \gamma'}(- \otimes \Theta_{\beta\gamma'})$  by  $\mathfrak{f}_{\alpha, \beta, \gamma'}$ . Setting  $\mathfrak{S}_{\mathfrak{h}} = \mathfrak{S}_1 + \mathfrak{S}_2$ , the study of different possible degenerations of a square to a bigon and a square or to two triangles gives

$$\begin{aligned} \mathfrak{f}_{\alpha, \beta, \gamma'} - \mathfrak{f}_{\mathbb{I}, \mathbb{S}, H'} \circ \Phi_{\mathfrak{h}} &= \mathfrak{S}_1 \circ d + d \circ \mathfrak{S}_1 \text{ and} \\ \Psi_{\mathfrak{h}} \circ \mathfrak{f}_{\mathbb{I}, \mathbb{S}, H} - \mathfrak{f}_{\alpha, \beta, \gamma'} &= \mathfrak{S}_2 \circ d + d \circ \mathfrak{S}_2 \\ \Rightarrow \Psi_{\mathfrak{h}} \circ \mathfrak{f}_{\mathbb{I}, \mathbb{S}, H} - \mathfrak{f}_{\mathbb{I}, \mathbb{S}, H'} \circ \Phi_{\mathfrak{h}} &= \mathfrak{S}_{\mathfrak{h}} \circ d + d \circ \mathfrak{S}_{\mathfrak{h}}. \end{aligned}$$

The square map  $\mathfrak{S}_{\mathfrak{h}}$  is trivial when  $\mathfrak{h}$  is a stabilization or destabilization, provided that the complex structure is sufficiently stretched along the neck. Changing the path of complex structures on the Heegaard diagram also corresponds to a square map, by standard argument in Floer theory.  $\square$

The above theorem justifies dropping the reference to the  $\mathbb{A}$ -diagram  $H$  in our notation for  $\mathfrak{f}_{\mathbb{I}, \mathbb{S}, t}$ . The map  $\mathfrak{f}_{\mathbb{I}, \mathbb{S}, t}$  gives a well-defined  $\mathbb{A}$ -homomorphism

$$\mathfrak{f}_{\mathbb{I}, \mathbb{S}, t}^{\mathbb{M}} : \mathrm{HF}^{\mathbb{M}}(\mathcal{T}) \longrightarrow \mathrm{HF}^{\mathbb{M}}(\mathcal{T}' = \mathcal{T}(\mathbb{I}, \mathbb{S}, t)),$$

for every  $\mathbb{A}$ -module  $\mathbb{M}$ . In fact, we may prove a slightly stronger form of independence from the framed arcs and framed spheres at this point. If  $\mathbb{I}'$  is obtained from  $\mathbb{I}$  by arc slides and  $\mathbb{S}'$  is obtained from  $\mathbb{S}$  by handle slides, then  $\mathcal{T}(\mathbb{I}', \mathbb{S}', t)$  may be identified with  $\mathcal{T}(\mathbb{I}, \mathbb{S}, t)$  (via an implicit diffeomorphism) and  $\mathcal{C}(\mathbb{I}', \mathbb{S}', t)$  may also be identified with  $\mathcal{C}(\mathbb{I}, \mathbb{S}, t)$  (via another implicit diffeomorphism, which is compatible with the first). We then obtain the maps

$$\mathfrak{f}_{\mathbb{I}, \mathbb{S}, t}^{\mathbb{M}}, \mathfrak{f}_{\mathbb{I}', \mathbb{S}', t}^{\mathbb{M}} : \mathrm{HF}^{\mathbb{M}}(\mathcal{T}) \longrightarrow \mathrm{HF}^{\mathbb{M}}(\mathcal{T}' = \mathcal{T}(\mathbb{I}, \mathbb{S}, t)).$$

**Theorem 6.12.** *Let  $\mathcal{C} = [W, F, \mathfrak{t}, \mathfrak{u}_F] = \mathcal{C}(\mathbb{I}, \mathbb{S}, \mathfrak{t})$  be an  $\mathbb{A}$ -cobordism defined by an acceptable set  $\mathbb{I}$  of framed arcs and a framed link  $\mathbb{S}$  for an  $\mathbb{A}$ -tangle  $\mathcal{T} = [M, T, \mathfrak{s}, \mathfrak{u}]$ . Suppose that  $\mathbb{I}'$  is obtained from  $\mathbb{I}$  by arc slides and  $\mathbb{S}'$  is obtained from  $\mathbb{S}$  by handle slides. Then*

$$d_* \circ \mathfrak{f}_{\mathbb{I}, \mathbb{S}, \mathfrak{t}}^{\mathbb{M}} = \mathfrak{f}_{\mathbb{I}', \mathbb{S}', \mathfrak{t}}^{\mathbb{M}} : \mathrm{HF}^{\mathbb{M}}(\mathcal{T}) \longrightarrow \mathrm{HF}^{\mathbb{M}}(\mathcal{T}''),$$

where  $d$  is the diffeomorphism from  $\mathcal{T}' = \mathcal{T}(\mathbb{I}, \mathbb{S}, \mathfrak{t})$  to  $\mathcal{T}'' = \mathcal{T}(\mathbb{I}', \mathbb{S}', \mathfrak{t})$ .

**Proof.** The proof is similar to the proof of Theorem 6.11. Thus, we will only provide an outline of the proof, and will leave checking the details to the reader. Let us assume that

$$H = (\Sigma, \boldsymbol{\alpha} = \{\alpha_1, \dots, \alpha_\ell\}, \boldsymbol{\beta} = \{\beta_1, \dots, \beta_\ell\}, \boldsymbol{\gamma} = \{\gamma_1, \dots, \gamma_\ell\}, \mathfrak{u} : \mathbf{z} \rightarrow \mathbb{A}, \mathfrak{t})$$

is a  $\mathbb{A}$ -diagram subordinate to the framed arcs  $\mathbb{I}$  and the framed link  $\mathbb{S}$ , as in Definition 6.7. We may then obtain a Heegaard diagram subordinate to  $\mathbb{I}'$  and  $\mathbb{S}'$  by doing handle-slides, supported away from the marked points, on  $\boldsymbol{\gamma}$  to obtain a set  $\boldsymbol{\delta}$  of  $\ell$  simple closed curves. Figure 7 illustrates how this is done for a single arc slide of  $\mathbb{I}_1$  over  $\mathbb{I}_2$ . If  $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}', \boldsymbol{\gamma}', \mathbf{z})$  is a Heegaard diagram subordinate to  $\{\mathbb{I}_1, \mathbb{I}_2\}$  which was constructed earlier, we may first do simultaneous handle slides on  $\boldsymbol{\beta}'$  and  $\boldsymbol{\gamma}'$  (i.e. handle slide  $\beta'_2$  and  $\gamma'_2$  over  $\beta'_1$  and  $\gamma'_1$ , respectively) to obtain a new diagram  $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \mathbf{z})$  subordinate to  $\{\mathbb{I}_1, \mathbb{I}_2\}$ . In the new diagram, if we do a handle slide among  $\boldsymbol{\gamma}$  (i.e. slide  $\gamma_2$  over  $\gamma_1$ ) we obtain a new set  $\boldsymbol{\delta}$  of simple closed curves so that  $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\delta}, \mathbf{z})$  is subordinate to  $\mathbb{I}'$ . The general case may be handles in a completely similar manner.

The  $\mathrm{Spin}^c$  class  $\mathfrak{t}$  in  $\mathrm{Spin}^c(W)$  determines a  $\mathrm{Spin}^c$  class on the 4-manifold  $W_H$ , which corresponds to the Heegaard quadruple  $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}, \mathbf{z})$  and associated with  $H = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}, \mathfrak{u}, \mathfrak{t})$  we obtain a holomorphic square map

$$\mathfrak{S} : \mathrm{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathfrak{u}, \mathfrak{s}) \rightarrow \mathrm{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\delta}, \mathfrak{u}, \mathfrak{s}'),$$

where  $\mathfrak{s}'$  is the restriction of  $\mathfrak{t}$  to  $\mathcal{T}'' = \mathcal{T}(\mathbb{I}', \mathbb{S}', \mathfrak{t})$ .

From degeneration arguments we also obtain the relation

$$\begin{aligned} d \circ \mathfrak{S} + \mathfrak{S} \circ d &= \mathfrak{f}_{\alpha\gamma\delta} \circ \mathfrak{f}_{\mathbb{I}, \mathbb{S}, \mathfrak{t}} - \mathfrak{f}_{\alpha, \beta, \delta} (- \otimes \mathfrak{f}_{\beta, \gamma, \delta} (\Theta_{\beta\gamma} \otimes \Theta_{\gamma\delta})) \\ &= \mathfrak{f}_{\alpha\gamma\delta} \circ \mathfrak{f}_{\mathbb{I}, \mathbb{S}, \mathfrak{t}} - \mathfrak{f}_{\alpha, \beta, \delta} (- \otimes \Theta_{\beta\delta}) \\ &= \mathfrak{f}_{\alpha\gamma\delta} \circ \mathfrak{f}_{\mathbb{I}, \mathbb{S}, \mathfrak{t}} - \mathfrak{f}_{\mathbb{I}', \mathbb{S}', \mathfrak{t}}. \end{aligned}$$

Note that  $\mathfrak{f}_{\alpha\gamma\delta}$  is the natural isomorphism which corresponds to the diffeomorphism  $d$  changing  $\mathcal{T}'$  to  $\mathcal{T}(\mathbb{I}', \mathbb{S})$ . This completes the proof of the desired equality.  $\square$

Combining Theorem 6.11 and Theorem 6.12, we conclude that for computing the map  $\mathfrak{f}_{\mathbb{I}, \mathbb{S}}^{\mathfrak{t}}$  we may always restrict ourselves to ordered sets of framed

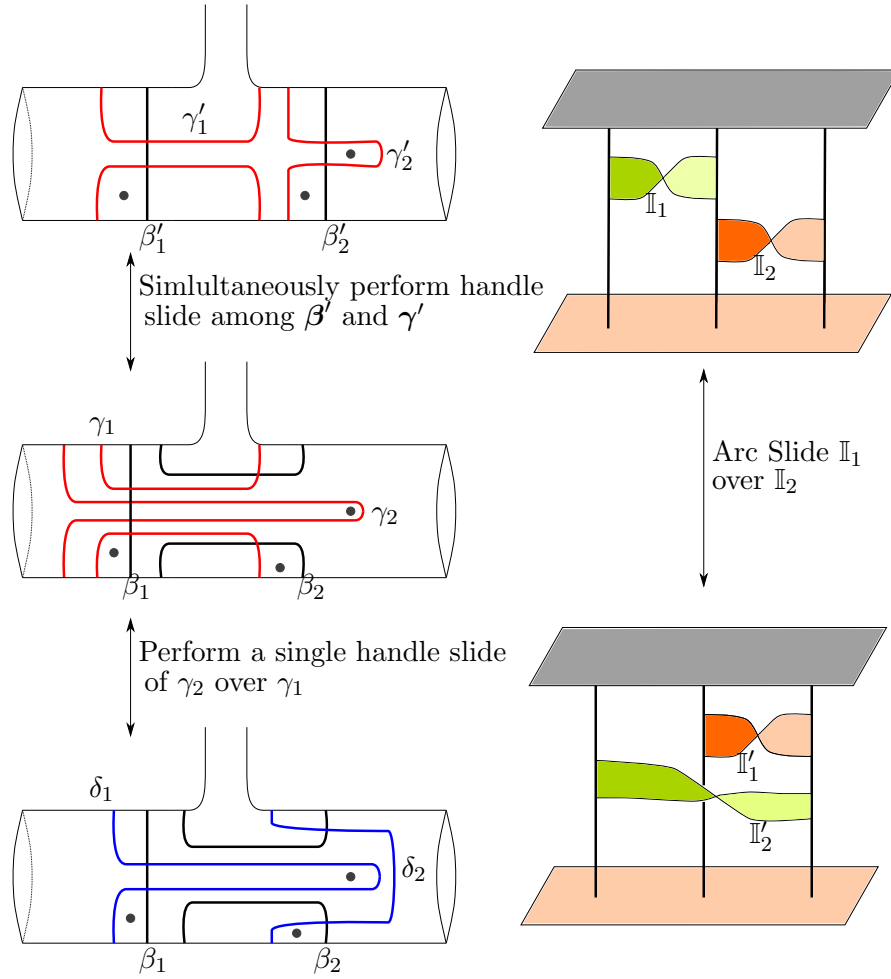


FIGURE 7. When we slide a framed arc  $\mathbb{I}_1$  over another framed arc  $\mathbb{I}_2$  to obtain  $\mathbb{I}' = \{\mathbb{I}'_1, \mathbb{I}'_2\}$  from  $\mathbb{I} = \{\mathbb{I}_1, \mathbb{I}_2\}$  we may choose a Heegaard diagram  $(\Sigma, \alpha, \beta, \delta, \gamma, \mathbf{z})$  so that the sub-diagrams  $(\Sigma, \alpha, \beta, \gamma, \mathbf{z})$  and  $(\Sigma, \alpha, \beta, \delta, \mathbf{z})$  are subordinate to  $\mathbb{I}$  and  $\mathbb{I}'$  respectively. For this purpose, we first need to perform a simultaneous handle-slide on a standard Heegaard diagram subordinate to  $\mathbb{I}$ .

arcs. This is particularly useful when we study the composition law in the following section.

## 7. THE COBORDISM MAP AND ITS INVARIANCE

**7.1. A composition law for framed arcs and links.** Let  $\mathcal{C} = [W, F, \mathfrak{t}, \mathfrak{u}_F]$  be an  $\mathbb{A}$ -cobordism from the  $\mathbb{A}$ -tangle  $\mathcal{T} = [M, T, \mathfrak{s}, \mathfrak{u}]$  to the  $\mathbb{A}$ -tangle  $\mathcal{T}' = [M', T', \mathfrak{s}', \mathfrak{u}']$ . Let us further assume that  $\mathcal{C}$  is an  $\mathbb{A}$ -cobordism associated with an acceptable set  $\mathbb{I}$  of framed arcs, which is a union  $\mathbb{I} = \mathbb{I}^1 \sqcup \mathbb{I}^2$ , and a framed link  $\mathbb{S}$ , which is also a union  $\mathbb{S} = \mathbb{S}^1 \sqcup \mathbb{S}^2$ . Denote the restriction of  $\mathfrak{t}$  to  $W(\mathbb{S}^1)$  by  $\mathfrak{t}^1$ . Suppose furthermore that  $\mathbb{I}^1$  is acceptable, as a set of framed arcs for  $\mathcal{T}$ . Let us denote  $\mathcal{T}(\mathbb{I}^1, \mathbb{S}^1, \mathfrak{t}^1)$  by  $\mathcal{T}''$ . Correspondingly, we obtain a set of framed arcs and a framed link in  $\mathcal{T}''$ , which are induced by  $\mathbb{I}^2$  and  $\mathbb{S}^2$ , respectively. We will abuse the notation and keep the names  $\mathbb{I}^2$  and  $\mathbb{S}^2$  for these induced objects. We also obtain a  $\text{Spin}^c$  structure on  $W(\mathbb{S}^2)$ , which is denoted by  $\mathfrak{t}^2$ , where  $\mathbb{S}^2$  refers to the framed link induced by  $\mathbb{S}^2$ . Consider the  $\mathbb{A}$ -cobordisms

$$\begin{aligned} \mathcal{C}^1 &= \mathcal{C}(\mathbb{I}^1, \mathbb{S}^1, \mathfrak{t}^1) = [W^1, F^1, \mathfrak{t}^1, \mathfrak{u}^1] : \mathcal{T} \rightsquigarrow \mathcal{T}'' \quad \text{and} \\ \mathcal{C}^2 &= \mathcal{C}(\mathbb{I}^2, \mathbb{S}^2, \mathfrak{t}^2) = [W^2, F^2, \mathfrak{t}^2, \mathfrak{u}^2] : \mathcal{T}'' \rightsquigarrow \mathcal{T}', \end{aligned}$$

which remain the same when we change  $\mathfrak{t}$  by elements of  $\delta H^1(M_{\mathcal{T}'}, \mathbb{Z})$ . Here

$$\delta : H^1(M_{\mathcal{T}'}, \mathbb{Z}) \longrightarrow H^2(W, \mathbb{Z})$$

is the connecting homomorphism in the Mayer-Vietoris sequence for  $(W^1, W^2)$ .

**Theorem 7.1.** *With the above notation fixed, for every  $\mathbb{A}$ -module  $\mathbb{M}$  we have*

$$\sum_{\substack{\mathfrak{t} \in \text{Spin}^c(W) \\ \mathfrak{t}|_{W^i} = \mathfrak{t}^i}} \mathfrak{f}_{\mathbb{I}, \mathbb{S}, \mathfrak{t}}^{\mathbb{M}} = \mathfrak{f}_{\mathbb{I}^2, \mathbb{S}^2, \mathfrak{t}^2}^{\mathbb{M}} \circ \mathfrak{f}_{\mathbb{I}^1, \mathbb{S}^1, \mathfrak{t}^1}^{\mathbb{M}}.$$

**Proof.** We may assume that  $\mathbb{I}$  is ordered as  $\mathbb{I} = (\mathbb{I}_1, \dots, \mathbb{I}_n)$ . Furthermore, since  $\mathbb{I}^1$  is acceptable, we may further assume that

$$\mathbb{I}^1 = (\mathbb{I}_1, \dots, \mathbb{I}_{n_1}) \quad \text{and} \quad \mathbb{I}^2 = (\mathbb{I}_{n_1+1}, \dots, \mathbb{I}_{n_1+n_2}), \quad \text{where } n_1 + n_2 = n.$$

Furthermore, suppose that

$$\mathbb{S}^1 = (\mathbb{S}_1, \dots, \mathbb{S}_{m_1}) \quad \text{and} \quad \mathbb{S}^2 = (\mathbb{S}_{m_1+1}, \dots, \mathbb{S}_{m_1+m_2}).$$

Let us fix an  $\mathbb{A}$ -diagram  $H = (\Sigma, \alpha, \beta, \gamma, \mathfrak{u}, \mathfrak{t})$  subordinate to the ordered set  $\mathbb{I}$  of framed arcs and the framed link  $\mathbb{S}$ . Let  $\delta = \{\delta_1, \dots, \delta_\ell\}$  denote a set of curves obtained as follows:

- Let  $\delta_i$  be a Hamiltonian isotope of  $\gamma_i$  for every  $i$  in

$$A = \{1, \dots, n_1\} \cup \{n_1 + 1, \dots, n_1 + m_1\}.$$

- For every  $i \in \{1, \dots, \ell\} - A$ , let  $\delta_i$  be a Hamiltonian isotope of  $\beta_i$ .

The  $\text{Spin}^c$  structures  $\mathfrak{t}^1$  and  $\mathfrak{t}^2$  induce  $\text{Spin}^c$  structures on  $W_{\alpha\beta\delta}$  and  $W_{\alpha\delta\gamma}$ , which will also be denoted by  $\mathfrak{t}^1$  and  $\mathfrak{t}^2$ , by slight abuse of notation. Moreover,  $\mathfrak{t}^1$  and  $\mathfrak{t}^2$  determine a coherent system of  $\text{Spin}^c$  structures on the Heegaard diagram  $(\Sigma, \alpha, \beta, \delta, \gamma, \mathbf{z})$  which will be referred to by  $\mathfrak{T}$ . The corresponding triangle classes for  $(\Sigma, \alpha, \beta, \gamma, \mathbf{z})$  are all classes corresponding to some  $\mathfrak{t} \in \text{Spin}^c(W)$  with  $\mathfrak{t}|_{W^i} = \mathfrak{t}^i$ . Then the  $\mathbb{A}$ -diagrams

$$H^1 = (\Sigma, \alpha, \beta, \delta, \mathbf{u}, \mathfrak{t}^1) \quad \text{and} \quad H^2 = (\Sigma, \alpha, \delta, \gamma, \mathbf{u}, \mathfrak{t}^2)$$

are subordinate to  $(\mathbb{I}^1, \mathbb{S}^1)$  and  $(\mathbb{I}^2, \mathbb{S}^2)$ , respectively. These two Heegaard triples determine the maps  $\mathfrak{f}^1 = \mathfrak{f}_{\mathbb{I}^1, \mathbb{S}^1, \mathfrak{t}^1}$  and  $\mathfrak{f}^2 = \mathfrak{f}_{\mathbb{I}^2, \mathbb{S}^2, \mathfrak{t}^2}$ . Furthermore, the  $\mathbb{A}$ -diagram

$$(\Sigma, \alpha, \beta, \delta, \gamma, \mathbf{u}, \mathfrak{T})$$

and the distinguished generators

$$\Theta_{\beta\delta} \in \text{CF}(\Sigma, \beta, \delta, \mathbf{u}, \mathfrak{t}^1|_{M_{\beta\delta}}) \quad \text{and} \quad \Theta_{\delta\gamma} \in \text{CF}(\Sigma, \delta, \gamma, \mathbf{u}, \mathfrak{t}^2|_{M_{\delta\gamma}})$$

determine a holomorphic square map

$$\mathfrak{S} : \text{CF}(\Sigma, \alpha, \beta, \mathbf{u}, \mathfrak{s}) = \text{CF}(\mathcal{T}) \longrightarrow \text{CF}(\Sigma, \alpha, \gamma, \mathbf{u}, \mathfrak{s}') = \text{CF}(\mathcal{T}').$$

Considering different possible degenerations of a square class of index 1 and applying a mild generalization of Theorem 8.16 from [OS2], we obtain the relation

$$\mathfrak{f}^2 \circ \mathfrak{f}^1 - \sum_{\substack{\mathfrak{t} \in \text{Spin}^c(W) \\ \mathfrak{t}|_{W^i} = \mathfrak{t}^i}} \mathfrak{f}_{\alpha, \beta, \gamma, \mathfrak{t}}(- \otimes \mathfrak{f}_{\beta, \delta, \gamma}(\Theta_{\beta\delta} \otimes \Theta_{\delta\gamma})) = \mathfrak{S} \circ d + d \circ \mathfrak{S}$$

Here  $\mathfrak{f}_{\alpha, \beta, \gamma, \mathfrak{t}}$  and  $\mathfrak{f}_{\beta, \delta, \gamma}$  are the holomorphic triangle maps associated with the Heegaard triples

$$(\Sigma, \alpha, \beta, \gamma, \mathbf{u}, \mathfrak{t}) \quad \text{and} \quad (\Sigma, \beta, \delta, \gamma, \mathbf{u}, \mathfrak{t}_{\beta\delta\gamma}),$$

respectively. Here,  $\mathfrak{t}_{\beta\delta\gamma}$  denotes a canonically determined  $\text{Spin}^c$  structure on the 4-manifold  $W_{\beta\delta\gamma}$ . In order to complete the proof, it thus suffices to show that

$$\mathfrak{f}_{\beta, \delta, \gamma}(\Theta_{\beta\delta} \otimes \Theta_{\delta\gamma}) = \Theta_{\beta\gamma}.$$

Let us denote the surfaces associated with  $\mathbb{I}^1$  and  $\mathbb{I}^2$  by  $F^1$  and  $F^2$ , respectively. After mildly changing the notation set in the proof of Proposition 6.4, associated with the ordered set  $(\mathbb{I}_1, \dots, \mathbb{I}_n)$  of framed arcs a sequence of Heegaard diagrams

$$H^i = (\Sigma^i, \beta^i = \{\beta_1, \dots, \beta_i\}, \gamma^i = \{\gamma_1, \dots, \gamma_i\}, \mathbf{u}^i, \mathfrak{s}_0^i), \quad i = 1, \dots, n$$

is constructed, so that  $H^n$  represents  $\mathcal{T}^+ = [M_F, T_F \cup \overline{T}_F, \mathfrak{s}_0, \mathbf{u}_F^+]$ . Furthermore, we may let  $\delta^n$  be a collection of curves obtained by performing a Hamiltonian isotopy over

$$\{\gamma_1, \dots, \gamma_{n_1}, \beta_{n_1+1}, \dots, \beta_{n=n_1+n_2}\}.$$

We may then complete  $\beta^n$  to  $\beta^{n+m}$  by taking the connected sum of the Heegaard surface with  $m$  genus-one surfaces corresponding to the framed

knots  $\mathbb{S}_1, \dots, \mathbb{S}_m$  and their stacks, and adding the meridian  $\beta_{n+i}$  of the framed knot  $\mathbb{S}_i$  to  $\beta^n$  for  $i = 1, \dots, m$ . Similarly, we complete  $\gamma^n$  to  $\gamma^{n+m}$  by adding the framing  $\gamma_{n+i}$  of  $\mathbb{S}_i$  to  $\gamma^n$ , for  $i = 1, \dots, m$ . We may assume that  $\delta^{n+m}$  is obtained from  $\delta^n$  by adding Hamiltonian isotopes of

$$\{\gamma_{n+1}, \dots, \gamma_{n+m_1}, \beta_{n+m_1+1}, \dots, \beta_{n+m}\}.$$

Finally,  $\ell - n - m$  one-handles are attached to  $\Sigma^{n+m}$  and  $\beta^{n+m}, \delta^{n+m}$  and  $\gamma^{n+m}$  are completed to  $\beta = \beta^\ell, \delta = \delta^\ell$  and  $\gamma = \gamma^\ell$  by adding triples  $(\beta_i, \delta_i, \gamma_i)$  of isotopic curves which represent the belts of the attached one-handles. This way, the Heegaard diagram

$$\overline{H} = \left( \Sigma = \Sigma^\ell, \beta, \delta, \gamma, \bar{u} : \mathbf{z} \cup \bar{\mathbf{z}} \rightarrow \mathbb{A}_F^+, \mathbf{t}_{\beta\delta\gamma} \right)$$

is obtained.

Let us now assume that  $\Theta \in \mathbb{T}_\beta \cap \mathbb{T}_\gamma$  is a generator which contributes to  $\mathbf{f}_{\beta,\gamma,\delta}(\Theta_{\beta\delta} \otimes \Theta_{\delta\gamma})$  through a triangle class

$$\Delta = \Delta^\ell \in \pi_2(\Theta_{\beta\delta}, \Theta_{\delta\gamma}, \Theta)$$

of Maslov index 0.  $\Sigma^\ell$  is obtained from  $\Sigma^{\ell-1}$  by attaching a 1-handle, and correspondingly, when the necks are sufficiently stretched, we obtain a triangle class  $\Delta^{\ell-1}$  on  $\Sigma^{\ell-1}$  and a class  $\Delta'$  on the attached one-handle which connects the top generators

$$c_\ell^+ \in \beta_\ell \cap \delta_\ell, \quad b_\ell^+ \in \delta_\ell \cap \gamma_\ell \quad \text{and} \quad d_\ell \in \beta_\ell \cap \gamma_\ell.$$

It also follows that

$$\mu(\Delta^\ell) = \mu(\Delta^{\ell-1}) - \epsilon(d_\ell),$$

where  $\epsilon(d_\ell)$  is 0 if  $d_\ell$  is the top intersection point  $d_\ell^+$ , and is 1 otherwise. We are thus forced to have  $d_\ell = d_\ell^+$ . It also follows from the argument of Proposition 5.1 and the second part of Lemma 5.3 that, if the necks are sufficiently stretched,  $\mathcal{M}(\Delta^\ell)$  may be identified with  $\mathcal{M}(\Delta^{\ell-1})$ . This argument may in fact be repeated again and again to show that the generator  $\Theta$  uses the top intersection points

$$d_i^+ \in \beta_i \cap \gamma_i, \quad \text{for } i = n + m + 1, \dots, \ell,$$

and that  $\mathcal{M}(\Delta^\ell)$  may be identified with  $\mathcal{M}(\Delta^{n+m})$ , for a triangle class over  $\Sigma^{n+m}$ .

Now  $\Sigma^{n+m} = \Sigma^{n+m-1} \# E$  where  $E$  is a surface of genus 1. If we stretch the connected sum neck, the complex structure on  $\Sigma^{n+m}$  converges to the join of complex structures on the  $\Sigma^{n+m-1} - \{w\}$  and  $E - \{w'\}$ . We also obtain a decomposition of  $\Delta^{n+m}$  to the triangle classes  $\Delta^{n+m-1}$  (on  $\Sigma^{n+m-1}$ ) and  $\Delta'$  (on  $E$ ). The generator  $\Theta$  is forced to use the unique intersection point  $d_{n+m} \in \beta_{n+m} \cap \gamma_{n+m}$ . From the choice of the intersection points in  $\Theta_{\beta\delta}$  and  $\Theta_{\delta\gamma}$  it also follows that  $\mu(\Delta^{n+m-1}) = \mu(\Delta^{n+m}) = 0$ . Furthermore, as the weak limit of a sequence of holomorphic curves in  $\mathcal{M}(\Delta^{n+m})$ , as the

neck is stretched, we obtain a degenerate holomorphic curve  $u^{n+m-1}$  in the 0-dimensional moduli space  $\mathcal{M}(\Delta^{n+m-1})$ , which has coefficient  $k$  at  $w$ . The holomorphic curve  $u^{n+m-1}$  determines a point  $\rho(u^{n+m-1})$  in  $\text{Sym}^k(\mathbb{D})$ . Let  $\Delta_k$  denote the union of all triangle classes  $\Delta'$  over the genus-one surface  $E$  with coefficient  $k$  over the marked point  $w'$ . The arguments of Section 12 of [Lip], and in particular Lemma 12.2, Proposition 12.4 (in fact, Proposition A.3), imply that in the aforementioned weak limit,  $u^{n+m-1}$  is paired with a degenerate curve  $v$  on  $E$ , which is the union of

$$E \times \rho(u^{n+m-1}) \subset E \times \mathbb{D}$$

and the unique holomorphic representative of  $\Delta_0$ , to produce the only possible curve in the weak limit. Every such weak limit may be perturbed to a holomorphic curve representing  $\Delta^{n+m}$ , giving an identification of  $\mathcal{M}(\Delta^{n+m})$  with  $\mathcal{M}(\Delta^{n+m-1})$ , if the connected sum neck is sufficiently long. Again, we may repeat the above argument to find an identification of  $\mathcal{M}(\Delta^\ell)$  with  $\mathcal{M}(\Delta^n)$ , provided that attaching the 1-handles and taking connected sum with surfaces of genus 1 is done using sufficiently stretched necks.

The surface  $\Sigma^n$  is obtained from  $\Sigma^{n-1}$  by attaching a 1-handle and moving two of the markings (which we denote by  $z, z' \in \mathbf{z}$ ) over the attached 1-handle. The curves  $\beta_n, \delta_n$  and  $\gamma_n$  are all isotopic to the belt circle of the attached 1-handle, while the positions of  $z$  and  $z'$  in the cylinder representing the neck is the position illustrated on the right-hand-side of Figure 8. Correspondingly, when the necks are sufficiently stretched, we obtain a triangles class  $\Delta^{n-1}$  on  $\Sigma^{n-1}$  and a class  $\Delta'$  on the attached one-handle which connects the top generators

$$c_n^+ \in \beta_n \cap \delta_n, \quad b_n^+ \in \delta_n \cap \gamma_n \quad \text{and} \quad d_n \in \beta_n \cap \gamma_n.$$

It also follows that

$$\mu(\Delta^n) = \mu(\Delta^{n-1}) - \epsilon(d_n),$$

where  $\epsilon(d_n)$  is 0 if  $d_n$  is the top intersection point  $d^+n$ , and is 1 otherwise. We are thus forced to have  $d_n = d_n^+$ . It also follows from the argument of Proposition 5.1 and the second part of Lemma 5.3 that, if the necks are sufficiently stretched,  $\mathcal{M}(\Delta^n)$  may be identified with  $\mathcal{M}(\Delta^{n-1})$ . This argument may in fact be repeated to show that the generator  $\Theta$  uses the top intersection points

$$d_i^+ \in \beta_i \cap \gamma_i, \quad \text{for } i = 1, \dots, n,$$

and that  $\mathcal{M}(\Delta^\ell)$  may be identified with  $\mathcal{M}(\Delta^1)$ , which in turn consists of a single point.

The above argument shows that  $\Theta$  is forced to be the generator  $\Theta_{\beta\gamma}$ , and that the total contribution of holomorphic triangles to the coefficient of  $\Theta$  in  $\mathfrak{f}_{\beta,\gamma,\delta}(\Theta_{\beta\delta} \otimes \Theta_{\delta\gamma})$  is 1. This completes the proof of the theorem.  $\square$

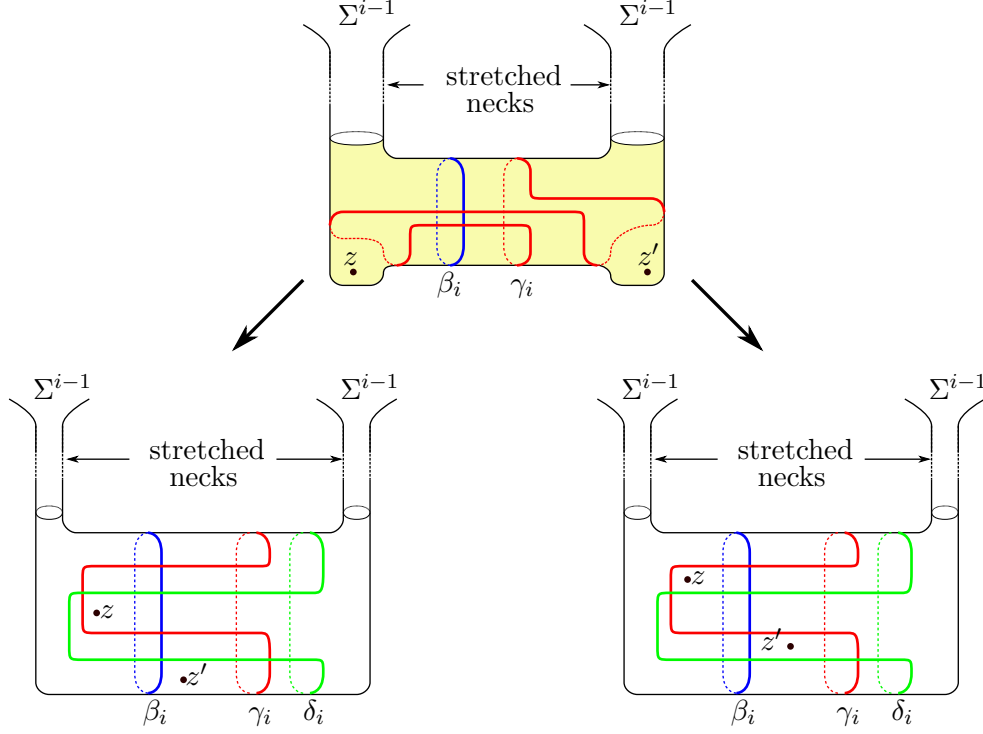


FIGURE 8. As we stretch the neck,  $\Sigma^i$  degenerates to a sub-surface  $\Sigma^{i-1}$  and a sphere, which are attached by two long necks. When  $i \in \{1, \dots, n_1\}$  a pair of markings  $z, z' \in \mathbf{z}$  will land on the sphere as illustrated on the left-hand-side, while for  $i \in \{n_1 + 1, \dots, n\}$  their location follows the pattern illustrated on the right-hand-side.

**7.2. The map associated with a cobordism.** Let  $\mathcal{C} = [W, F, \mathbf{t}, \mathbf{u}_F]$  be a stable  $\mathbb{A}$ -cobordism from  $\mathcal{T} = [M, T, \mathbf{s}, \mathbf{u}]$  to  $\mathcal{T}' = [M', T', \mathbf{s}', \mathbf{u}']$ . Consider an indefinite, ordered good Morse datum  $\mathfrak{M} = (G, b, \xi)$  for  $\mathcal{C}$  (or more precisely, for  $\mathcal{W} = (W, F)$ ). The Morse datum  $\mathfrak{M}$  induces a parametrized Cerf decomposition

$$\mathcal{C} = \mathcal{C}_1 \cup_{\mathcal{T}_1} \mathcal{C}_2 \cup_{\mathcal{T}_2} \mathcal{C}_3.$$

of  $\mathcal{C}$  where  $\mathcal{T}_i = [M_i, T_i, \mathbf{s}_i, \mathbf{u}_i]$  for  $i = 0, 1, 2, 3$  and  $\mathcal{T}_0 = \mathcal{T}$  and  $\mathcal{T}_3 = \mathcal{T}'$ . Moreover, let  $\mathcal{C}_i = [W_i, F_i, \mathbf{t}_i, \mathbf{u}_i]$  for  $i = 1, 2, 3$ , where  $\mathcal{C}_1, \mathcal{C}_2$  and  $\mathcal{C}_3$  are parametrized by a set  $\mathbb{S}^0 \subset \mathcal{T}_0$  of framed 0-spheres, a the sets  $\mathbb{I}$  and  $\mathbb{S}$  of framed arc and knots in  $\mathcal{T}_1$  and a set  $\mathbb{S}^2$  of framed 2-spheres in  $\mathcal{T}_2$ , respectively. Let

$$d_1 : \mathcal{T}_0(\mathbb{S}^0) \rightarrow \mathcal{T}_1, \quad d_2 : \mathcal{T}_1(\mathbb{I}, \mathbb{S}, \mathbf{t}_2) \rightarrow \mathcal{T}_2 \quad \text{and} \quad d_3 : \mathcal{T}_2(\mathbb{S}^2) \rightarrow \mathcal{T}_3$$

represent the isotopy classes of the corresponding diffeomorphisms. Note that  $\mathbf{t}$  determines  $\mathbf{t}_i$  for  $i = 1, 2, 3$ , while  $\mathbf{t}_2$  determines  $\mathbf{t}$ .



For every  $\mathbb{A}$ -module  $\mathbb{M}$ , the constructions of Section 5 and Section 6 associate naturally defined  $\mathbb{A}$ -homomorphisms

$$\mathfrak{f}_{\mathcal{C}_i, \mathfrak{M}}^{\mathbb{M}} = \text{HF}^{\mathbb{M}}(d_i) \circ \mathfrak{f}_{\mathbb{S}^i}^{\mathbb{M}} : \text{HF}^{\mathbb{M}}(\mathcal{T}_{i-1}) \longrightarrow \text{HF}^{\mathbb{M}}(\mathcal{T}_i)$$

to the  $\mathbb{A}$ -cobordism  $\mathcal{C}_i$  for  $i = 1, 3$  and

$$\mathfrak{f}_{\mathcal{C}_2, \mathfrak{M}}^{\mathbb{M}} = \text{HF}^{\mathbb{M}}(d_2) \circ \mathfrak{f}_{\mathbb{I}, \mathbb{S}, \mathfrak{t}_2}^{\mathbb{M}} : \text{HF}^{\mathbb{M}}(\mathcal{T}_1) \longrightarrow \text{HF}^{\mathbb{M}}(\mathcal{T}_2)$$

to the  $\mathbb{A}$ -cobordism  $\mathcal{C}_2$ . Subsequently, we may define

$$(4) \quad \begin{aligned} \mathfrak{f}_{\mathcal{C}, \mathfrak{M}}^{\mathbb{M}} &: \text{HF}^{\mathbb{M}}(\mathcal{T}) \longrightarrow \text{HF}^{\mathbb{M}}(\mathcal{T}') \\ \mathfrak{f}_{\mathcal{C}, \mathfrak{M}}^{\mathbb{M}} &= \mathfrak{f}_{\mathcal{C}_3, \mathfrak{M}}^{\mathbb{M}} \circ \mathfrak{f}_{\mathcal{C}_2, \mathfrak{M}}^{\mathbb{M}} \circ \mathfrak{f}_{\mathcal{C}_1, \mathfrak{M}}^{\mathbb{M}}. \end{aligned}$$

The homomorphism  $\mathfrak{f}_{\mathcal{C}, \mathfrak{M}}^{\mathbb{M}}$  is well-defined and natural, while *a priori* it depends on the Morse datum  $\mathfrak{M}$ .

**Theorem 7.2.** *Let  $\mathcal{C}$  be an  $\mathbb{A}$ -cobordism from the  $\mathbb{A}$ -tangle  $\mathcal{T}$  to the  $\mathbb{A}$ -tangle  $\mathcal{T}'$ . For every  $\mathbb{A}$ -module  $\mathbb{M}$ , the  $\mathbb{A}$ -homomorphism*

$$\mathfrak{f}_{\mathcal{C}, \mathfrak{M}}^{\mathbb{M}} : \text{HF}^{\mathbb{M}}(\mathcal{T}) \rightarrow \text{HF}^{\mathbb{M}}(\mathcal{T}')$$

*is an invariant of  $\mathcal{C}$ . More precisely, this  $\mathbb{A}$ -homomorphism does not depend on the choice of the indefinite, ordered good Morse datum  $\mathfrak{M} = (G, \underline{b}, \xi)$  for  $\mathcal{C}$ , which was used in its definition.*

**Proof.** Theorem 4.19 reduces the proof of the above theorem to showing the invariance of the chain map associated with an indefinite, ordered good Morse datum  $\mathfrak{M} = (G, \underline{b}, \xi)$  under the following changes:

- (1) Adding or removing regular values.
- (2) Creation and cancellation of critical points of  $G$ .
- (3) Switching two index-1, 2 or index 3 critical points of  $G$ .
- (4) Isotopies of the gradient-like vector field.
- (5) Left-right equivalences.
- (6) Switching two critical points of  $g = G|_F$ .
- (7) Switching an index-2 critical point of  $G$  with a critical point of its restriction  $g$  to  $F$ .

The proof of the invariance under the first 6 moves in the above list follows from Theorem 6.12 and by copying the arguments of Subsection 4.4 from [OS4]. Theorem 7.2 is thus reduced to proving the following proposition.

**Proposition 7.3.** *Suppose that the  $\mathbb{A}$ -cobordism  $\mathcal{C} = [W, F, \mathfrak{t}, \mathfrak{u}_F]$  from an  $\mathbb{A}$ -tangle  $\mathcal{T}$  to another  $\mathbb{A}$ -tangle  $\mathcal{T}'$  is determined by an acceptable set  $\mathbb{I}$  of framed arcs and a framed link  $\mathbb{S}$  in  $\mathcal{T}$ . Let  $\mathbb{I}'$  be the acceptable set of framed arcs obtained from  $\mathbb{I}$  by sliding some of the framed arcs over some components of the framed link  $\mathbb{S}$  and  $\mathbb{S}'$  be the framed link obtained from  $\mathbb{S}$  by sliding some of its components over some of the framed arcs in  $\mathbb{I}$ . Set*

$$\mathcal{C}_1 = \mathcal{C}(\mathbb{I}', \mathbb{S}, \mathfrak{t}) \quad \text{and} \quad \mathcal{C}_2 = \mathcal{C}(\mathbb{I}, \mathbb{S}', \mathfrak{t})$$

and let

$$d_1 : \mathcal{T}(\mathbb{I}, \mathbb{S}, \mathfrak{t}) \rightarrow \mathcal{T}(\mathbb{I}', \mathbb{S}, \mathfrak{t}) \quad \text{and} \quad d_2 : \mathcal{T}(\mathbb{I}, \mathbb{S}, \mathfrak{t}) \rightarrow \mathcal{T}(\mathbb{I}, \mathbb{S}', \mathfrak{t})$$

denote the induced diffeomorphisms. Then for every  $\mathbb{A}$ -module  $\mathbb{M}$  we have

$$\mathfrak{f}_{\mathcal{C}_1}^{\mathbb{M}} = \text{HF}^{\mathbb{M}}(d_1) \circ \mathfrak{f}_{\mathcal{C}}^{\mathbb{M}} \quad \text{and} \quad \mathfrak{f}_{\mathcal{C}_1}^{\mathbb{M}} = \text{HF}^{\mathbb{M}}(d_1) \circ \mathfrak{f}_{\mathcal{C}}^{\mathbb{M}}.$$

However, both claims in Proposition 7.3 are immediate consequences of Theorem 7.1. This completes the proof of Theorem 7.2  $\square$

Given Theorem 7.2, we may now denote the  $\mathbb{A}$ -homomorphism associated with the  $\mathbb{A}$ -cobordism  $\mathcal{C}$  and the  $\mathbb{A}$ -module  $\mathbb{M}$  by  $\mathfrak{f}_{\mathcal{C}}^{\mathbb{M}}$ .

**7.3. The composition law.** In this subsection, we will prove a generalization of Theorem 7.1.

**Theorem 7.4.** *Suppose that  $\mathcal{C}_i = [W_i, F_i, \mathfrak{t}_i, \mathfrak{u}_i] : \mathcal{T}_{i-1} \rightarrow \mathcal{T}_i$  are  $\mathbb{A}$ -cobordisms for  $i = 1, \dots, m$ . Let  $(W, F)$  be the stable cobordism obtained by putting  $(W_i, F_i)$  together, and  $\mathfrak{u} : \pi_0(F) \rightarrow \mathbb{A}$  denote the representation induced by  $\mathfrak{u}_i$ . For every  $\mathfrak{t} \in \text{Spin}^c(W)$  with  $\mathfrak{t}|_{W_i} = \mathfrak{t}_i$  let  $\mathcal{C}_{\mathfrak{t}}$  denote the  $\mathbb{A}$ -cobordism  $[W, F, \mathfrak{t}, \mathfrak{u}]$ . Then*

$$(5) \quad \sum_{\substack{\mathfrak{t} \in \text{Spin}^c(W) \\ \mathfrak{t}|_{W_i} = \mathfrak{t}_i}} \mathfrak{f}_{\mathcal{C}_{\mathfrak{t}}}^{\mathbb{M}} = \mathfrak{f}_{\mathcal{C}_m}^{\mathbb{M}} \circ \dots \circ \mathfrak{f}_{\mathcal{C}_1}^{\mathbb{M}}.$$

**Proof.** The definition of the cobordism invariants and Theorem 7.1 reduce the proof of Theorem 7.4 to the case where  $m = 2$ , both  $\mathcal{C}_1$  and  $\mathcal{C}_2$  are elementary, and one of the following happens.

- (1)  $\mathcal{C}_1$  corresponds to a framed  $S^2$  and  $\mathcal{C}_2$  corresponds to a framed  $S^0$ .
- (2)  $\mathcal{C}_1$  corresponds to a framed  $S^2$  and  $\mathcal{C}_2$  corresponds to a framed  $S^1$ .
- (3)  $\mathcal{C}_1$  corresponds to a framed  $S^2$  and  $\mathcal{C}_2$  corresponds to a framed arc.
- (4)  $\mathcal{C}_1$  corresponds to a framed  $S^1$  and  $\mathcal{C}_2$  corresponds to a framed  $S^0$ .
- (5)  $\mathcal{C}_1$  corresponds to a framed arc and  $\mathcal{C}_2$  corresponds to a framed  $S^0$ .

Let us assume that  $\mathcal{T} = \mathcal{T}_0 = [M, T, \mathfrak{s}, \mathfrak{u}]$  and  $\mathcal{T}' = \mathcal{T}_2 = [M', T', \mathfrak{s}', \mathfrak{u}']$ . In the first case, there is a 2-sphere  $\mathbb{S}$  and points  $x, y$  in  $M - T$  such that  $\mathcal{C}_1$  is obtained by attaching a 3-handle along  $\mathbb{S}$  and  $\mathcal{C}_2$  is obtained by attaching a 1-handle at  $x$  and  $y$ . Note that  $\langle c_1(\mathfrak{s}), [\mathbb{S}] \rangle = 0$  and  $\mathfrak{s}$  determines the  $\text{Spin}^c$  structure  $\mathfrak{t}_1$  on  $\mathcal{C}_1$ . Furthermore,  $\mathfrak{s}_1 = \mathfrak{t}_1|_{M_1}$  determines the  $\text{Spin}^c$  structure  $\mathfrak{t}_2$  on  $\mathcal{C}_2$ , and thus its restriction  $\mathfrak{s}'$  to  $M'$ . Let  $(W, F)$  denote the stable cobordism obtained by putting the underlying cobordisms of  $\mathcal{C}_1$  and  $\mathcal{C}_2$  together and  $\mathfrak{t} \in \text{Spin}^c(W)$  denote a  $\text{Spin}^c$  structure on  $W$  which restricts to  $\mathfrak{s}$  on  $M$  and thus  $\mathfrak{s}'$  on  $M'$ . Note that  $\mathfrak{s}$  determines  $\mathfrak{t}$ . Let  $\mathcal{C}_1^{\mathfrak{t}}$  denote the cobordism from  $\mathcal{T}$  to  $\mathcal{T}^{\mathfrak{t}}$  obtained by attaching a 1-handle to  $x, y$  and restricting  $\mathfrak{t}$  to the resulting stable cobordism, and  $\mathcal{C}_2^{\mathfrak{t}}$  denote the cobordism obtained by attaching a 3-handle to  $\mathbb{S}$  in  $\mathcal{T}^{\mathfrak{t}}$  and restricting  $\mathfrak{t}$  to the resulting stable cobordism. Note that the other vertical boundary component of  $\mathcal{C}_2^{\mathfrak{t}}$

is naturally identified with  $\mathcal{T}'$ . The  $\mathbb{A}$ -cobordism  $\mathcal{C}$  is then diffeomorphic to  $\mathcal{C}_1^t \cup_{\mathcal{T}^t} \mathcal{C}_2^t$  via a diffeomorphism which is the identity on the boundary. Consider the following diagram.

$$\begin{array}{ccc} \mathrm{HF}^{\mathbb{M}}(\mathcal{T}) & \xrightarrow{\mathfrak{f}_{\mathcal{C}_1}^{\mathbb{M}}} & \mathrm{HF}^{\mathbb{M}}(\mathcal{T}_1) \\ \mathfrak{f}_{\mathcal{C}_1^t}^{\mathbb{M}} \downarrow & & \downarrow \mathfrak{f}_{\mathcal{C}_2}^{\mathbb{M}} \\ \mathrm{HF}^{\mathbb{M}}(\mathcal{T}^t) & \xrightarrow{\mathfrak{f}_{\mathcal{C}_2^t}^{\mathbb{M}}} & \mathrm{HF}^{\mathbb{M}}(\mathcal{T}'). \end{array}$$

Every generator of  $\mathrm{HF}^{\mathbb{M}}(\mathcal{T})$  is of the form  $\mathbf{x} \times \theta$ , where  $\mathbf{x}$  is a generator of the chain complex giving  $\mathrm{HF}^{\mathbb{M}}(\mathcal{T}_1)$  and  $\theta$  is one of the two intersection points  $\theta_{top}$  and  $\theta_{bottom}$  corresponding to the framed  $S^2$ . If the necks in the corresponding Heegaard diagrams for the frames  $S^2$  and the framed  $S^0$  are sufficiently stretched, this generator is mapped to  $\mathbf{x} \times \theta \times \theta'_{top}$  under  $\mathfrak{f}_{\mathcal{C}_1}^{\mathbb{M}}$ . Here  $\theta'_{top}$  and  $\theta'_{bottom}$  are the two intersection points which correspond to the 1-handle, which is attached to  $x$  and  $y$ . The generator  $\mathbf{x} \times \theta'_{top}$  goes to zero under  $\mathfrak{f}_{\mathcal{C}_2}^{\mathbb{M}}$  unless  $\theta = \theta_{bottom}$ , when it is mapped to  $\mathbf{x} \times \theta'_{top}$ . On the other hand, the image of  $\mathbf{x} \times \theta$  under  $\mathfrak{f}_{\mathcal{C}_1}^{\mathbb{M}}$  is zero unless  $\theta = \theta_{bottom}$ , when

$$\mathfrak{f}_{\mathcal{C}_1}^{\mathbb{M}}(\mathbf{x} \times \theta) = \mathbf{x} \quad \text{and} \quad \mathfrak{f}_{\mathcal{C}_2}^{\mathbb{M}}(\mathfrak{f}_{\mathcal{C}_1}^{\mathbb{M}}(\mathbf{x} \times \theta)) = \mathbf{x} \times \theta'_{top}.$$

This implies that

$$\mathfrak{f}_{\mathcal{C}_2}^{\mathbb{M}} \circ \mathfrak{f}_{\mathcal{C}_1}^{\mathbb{M}} = \mathfrak{f}_{\mathcal{C}}^{\mathbb{M}}.$$

A similar modification to the proofs of Proposition 4.19 and Proposition 4.18 from [OS4], imply the Theorem in the second and fourth cases, respectively. The only remaining cases are thus the third and the fifth case.

In the third case,  $\mathcal{C}_1$  corresponds to attaching a 3-handle along some framed  $S^2$  in  $M - T$ , which may be denoted by  $\mathbb{S}$ , and  $\mathcal{C}_2$  corresponds to a framed arc  $\mathbb{I}$ . This framed arc may clearly be isotoped to  $M - \mathbb{S}$ . The  $\mathrm{Spin}^c$  structure  $\mathfrak{t}_1$  on  $\mathcal{C}_1$  determines a  $\mathrm{Spin}^c$  structure  $\mathfrak{t}$  on  $W$ . Correspondingly, we may choose a Heegaard surface  $\Sigma$  in  $M$  with the following properties:

- The neighborhood of the intersection of the Heegaard surface  $\Sigma$  with  $\mathbb{S}$  is a cylinder  $S$  with boundary components  $C_1$  and  $C_2$ . This surface corresponds to the surface  $\Sigma^\circ$  in  $M_1 = M_{\mathcal{T}_1}$ , which is obtained by cutting off  $S$  and gluing a pair of disks to  $C_1$  and  $C_2$ . Moreover, the surface  $\Sigma$  cuts  $T$  in  $\mathbf{z}$ .
- There are two collections  $\alpha$  and  $\beta$  of attaching curves on  $\Sigma$  such that  $\alpha \cap S$  is a single curve  $\alpha$  which is a small Hamiltonian isotope of  $\{\beta\} = \beta \cap S$ . Moreover,  $(\Sigma, \alpha, \beta, \mathbf{u} : \mathbf{z} \rightarrow \mathbb{A}, \mathfrak{s})$  corresponds to  $\mathcal{T}$  and

$$(\Sigma^\circ, \alpha - \{\alpha\}, \beta - \{\beta\}, \mathbf{u} : \mathbf{z} \rightarrow \mathbb{A}, \mathfrak{s}_1)$$

corresponds to  $\mathcal{T}_1$ .

- There is third collection  $\gamma$  of attaching curves on  $\Sigma$  so that  $\gamma \cap S$  is a single curve  $\gamma$  which is a small Hamiltonian isotope of both  $\alpha$  and  $\beta$  and so that

$$(\Sigma^\circ, \alpha - \{\alpha\}, \beta - \{\beta\}, \gamma - \{\gamma\}, \mathbf{z})$$

is subordinate to the framed arc  $\mathbb{I}$ , and in particular, the diagram  $(\Sigma^\circ, \alpha - \{\alpha\}, \gamma - \{\gamma\}, \mathbf{z})$  is a tangle diagram for  $(M_2, T_2)$ .

Let  $\hat{\alpha} = \alpha - \{\alpha\}$ ,  $\hat{\beta} = \beta - \{\beta\}$  and  $\hat{\gamma} = \gamma - \{\gamma\}$ . Let  $\mathcal{C}'_1$  denote the elementary  $\mathbb{A}$ -cobordism from  $\mathcal{T}$  to the  $\mathbb{A}$ -tangle  $\mathcal{T}'_1$  determined by  $\mathbb{I}$ . The diagram  $(\Sigma, \alpha, \beta, \gamma, \mathbf{u}, \mathbf{t}'_1)$  is then subordinate to  $\mathcal{C}'_1$ , where  $\mathbf{t}'_1$  is determined by  $\mathfrak{s} = \mathbf{t}|_{\mathcal{T}}$ . Let us assume that the path of almost complex structures is sufficiently stretched along  $C_1$  and  $C_2$ . Consider the diagram

$$\begin{array}{ccc} \mathrm{CF}(\Sigma, \alpha, \beta, \mathbf{u}, \mathbf{t}|_{\mathcal{T}}) & \xrightarrow{\mathfrak{f}_{\mathcal{C}_1}} & \mathrm{CF}(\Sigma^\circ, \hat{\alpha}, \hat{\beta}, \mathbf{u}, \mathbf{t}|_{\mathcal{T}_1}) \\ \mathfrak{f}_{\mathcal{C}'_1} \downarrow & & \downarrow \mathfrak{f}_{\mathcal{C}_2} \\ \mathrm{CF}(\Sigma, \alpha, \gamma, \mathbf{u}, \mathbf{t}|_{\mathcal{T}'_1}) & \xrightarrow{\mathfrak{f}_{\mathcal{C}'_2}} & \mathrm{CF}(\Sigma^\circ, \hat{\alpha}, \hat{\gamma}, \mathbf{u}, \mathbf{t}|_{\mathcal{T}'}) \end{array}$$

Every generator of  $\mathrm{CF}(\Sigma, \alpha, \beta, \mathbf{u}, \mathbf{t}|_{\mathcal{T}})$  is of the form  $\mathbf{x} \times \theta$ , where  $\mathbf{x}$  is a generator of  $\mathrm{CF}(\Sigma^\circ, \hat{\alpha}, \hat{\beta}, \mathbf{u}, \mathbf{t}|_{\mathcal{T}_1})$  and  $\theta$  is one of the two intersection points  $\theta_{top}$  and  $\theta_{bottom}$  are the intersection points between  $\alpha$  and  $\beta$  which correspond to  $\mathbb{S}$ . The image of such a generator under  $\mathfrak{f}_{\mathcal{C}_1}$  is trivial unless  $\theta = \theta_{bottom}$ , when we have

$$\mathfrak{f}_{\mathcal{C}_1}(\mathbf{x} \times \theta) = \mathbf{x}.$$

On the other hand, we may use the argument of Theorem 7.1 and show that

$$\mathfrak{f}_{\mathcal{C}'_1}(\mathbf{x} \times \theta) = \mathfrak{f}_{\mathcal{C}_2}(\mathbf{x}) \times \theta'$$

where  $\theta' \in \alpha \cap \gamma$  is the intersection point which corresponds to  $\theta$ , and is one of  $\theta'_{top}$  and  $\theta'_{bottom}$ . Every such generator is mapped to 0 by  $\mathfrak{f}_{\mathcal{C}'_2}$  unless  $\theta' = \theta'_{bottom}$ , or equivalently, unless  $\theta = \theta_{bottom}$ . If this condition is satisfied, then

$$\mathfrak{f}_{\mathcal{C}'_2}(\mathfrak{f}_{\mathcal{C}'_1}(\mathbf{x} \times \theta)) = \mathfrak{f}_{\mathcal{C}_2}(\mathbf{x}),$$

which completes the proof of the third case. The proof of the fifth case is very similar to the proof of the third case.  $\square$

The composition law of Theorem 7.4 implies, in particular, that the left-hand-side expression is well-defined. Let us define

$$W^0 = \bigcup_{i \equiv 0 \pmod{2}} W_i \quad \text{and} \quad W^1 = \bigcup_{i \equiv 1 \pmod{2}} W_i.$$

Then  $W^0 \cup W^1 = W$  and  $W^0 \cap W^1 = \cup_{i=1}^{m-1} M_i$ , where  $M_i = M_{\mathcal{T}_i}$ , and we obtain the following cohomology long exact sequence

$$\dots \longrightarrow \bigoplus_{i=1}^{m-1} H^1(M_i, \mathbb{Z}) \xrightarrow{\delta} H^2(W) \xrightarrow{\pi} \bigoplus_{i=1}^m H^2(W_i, \mathbb{Z}) \longrightarrow \dots$$

If  $\mathfrak{t}, \mathfrak{t}' \in \text{Spin}^c(W)$  restrict to  $\mathfrak{t}_i$  on  $W_i$ , then  $\mathfrak{t} - \mathfrak{t}' \in H^2(W, \mathbb{Z})$  is a class in the kernel of  $\pi$ , and is thus in the image of  $\delta$ . In particular, the subset  $\mathfrak{T} \subset \text{Spin}^c(W)$  which appears in the summation of the left-hand-side of Equation 5 is the orbit of a fixed  $\text{Spin}^c$  structure  $\mathfrak{t}$  under the action of the  $\mathbb{Z}$ -module  $\text{Im}(\delta)$  of  $H^2(W, \mathbb{Z})$ . On the other hand, using an appropriate Morse datum  $\mathfrak{M}$ , we may represent every  $\mathbb{Z}$ -submodule of  $H^2(W, \mathbb{Z})$  as  $\text{Im}(\delta)$  for some decomposition of the stable cobordism  $(W, F)$ . In particular, for every affine set  $\mathfrak{T}$  of  $\text{Spin}^c$  structures over a  $\mathbb{Z}$ -submodule of  $H^2(W, \mathbb{Z})$  which restrict to  $\mathfrak{s}$  and  $\mathfrak{s}'$  on the two ends, the sum

$$\sum_{\mathfrak{t} \in \mathfrak{T}} \mathfrak{f}_{\mathcal{C}_{\mathfrak{t}}}^{\mathfrak{M}} : \text{HF}^{\mathfrak{M}}(\mathcal{T}) \rightarrow \text{HF}^{\mathfrak{M}}(\mathcal{T}')$$

is well-defined.

**Definition 7.5.** Let  $\mathcal{C} = [W, F, \mathfrak{T}, \mathfrak{u}_F] : \mathcal{T} \rightsquigarrow \mathcal{T}'$  be an arbitrary  $\mathbb{A}$ -cobordism from the  $\mathbb{A}$ -tangle  $\mathcal{T}$  to the  $\mathbb{A}$ -tangle  $\mathcal{T}'$ , where  $\mathfrak{T}$  is a subset of  $\text{Spin}^c(W)$  which is affine over a  $\mathbb{Z}$ -submodule of  $H^2(W, \mathbb{Z})$ . For every  $\mathfrak{t} \in \mathfrak{T}$  define  $\mathcal{C}_{\mathfrak{t}} = [W, F, \mathfrak{t}, \mathfrak{u}]$ . We then define the cobordism map associated with  $\mathcal{C}$  by

$$\mathfrak{f}_{\mathcal{C}}^{\mathfrak{M}} := \sum_{\mathfrak{t} \in \mathfrak{T}} \mathfrak{f}_{\mathcal{C}_{\mathfrak{t}}}^{\mathfrak{M}}.$$

With the above definition in place, we may then re-state Theorem 7.2 and Theorem 7.1 as the following theorem.

**Theorem 7.6.** Fix an algebra  $\mathbb{A}$  over  $\mathbb{F}$  and a  $\mathbb{A}$ -module  $\mathfrak{M}$ . Assigning the  $\mathbb{A}$ -module  $\text{HF}^{\mathfrak{M}}(\mathcal{T})$  to every  $\mathbb{A}$ -tangle  $\mathcal{T} \in \text{Obj}(\mathbb{A}\text{-Tangles})$  and the  $\mathbb{A}$ -homomorphism  $\mathfrak{f}_{\mathcal{C}}^{\mathfrak{M}} : \text{HF}^{\mathfrak{M}}(\mathcal{T}) \rightarrow \text{HF}^{\mathfrak{M}}(\mathcal{T}')$  to every  $\mathbb{A}$ -cobordism

$$(\mathcal{C} : \mathcal{T} \rightsquigarrow \mathcal{T}') \in \text{Mor}(\mathcal{T}, \mathcal{T}') \in \text{Mor}(\mathbb{A}\text{-Tangles})$$

gives a well-defined functor

$$\text{HF}^{\mathfrak{M}} : \mathbb{A}\text{-Tangles} \longrightarrow \mathbb{A}\text{-Modules}.$$

**7.4. Action of  $\Lambda^*(H_1(W, \mathbb{Z})/\text{Tors})$ .** Let us assume that  $\mathcal{C} = [W, F, \mathfrak{t}, \mathfrak{u}_F]$  is an  $\mathbb{A}$ -cobordism from  $\mathcal{T} = [M, T, \mathfrak{s}, \mathfrak{u}]$  to  $\mathcal{T}' = [M', T', \mathfrak{s}', \mathfrak{u}']$ . We may then decompose  $\mathcal{C}$  as

$$\mathcal{C} = \mathcal{C}_1 \cup_{\mathcal{T}_1} \mathcal{C}_2 \cup_{\mathcal{T}_2} \mathcal{C}_3$$

where  $\mathcal{C}_1$  corresponds to the addition of 1-handles,  $\mathcal{C}_2$  corresponds to the addition of 2-handles along framed links and band surgeries along framed arcs, and  $\mathcal{C}_3$  corresponds to the addition of 3-handles. Let  $\mathcal{C}' = \mathcal{C}_2 = [W', F', \mathfrak{t}', \mathfrak{u}_{F'}]$ . It is then clear that  $H_1(W', \mathbb{Z}) = H_1(W, \mathbb{Z})$ .

The  $\mathbb{A}$ -cobordism  $\mathcal{C}'$  is of the form  $\mathcal{C}' = \mathcal{C}(\mathbb{S}, \mathbb{I}, \mathbf{t}')$  where  $\mathbb{S}$  is a framed link and  $\mathbb{I}$  is an acceptable set of framed arcs. In order to define  $\mathfrak{f}_{\mathcal{C}'}^{\mathbb{M}}$  we may thus use an  $\mathbb{A}$ -diagram

$$H = (\Sigma, \alpha, \beta, \gamma, \mathbf{u}, \mathbf{t})$$

subordinate to  $\mathbb{I}$  and  $\mathbb{S}$ . Let us denote by  $W_{\alpha\beta\gamma}$  the 4-manifold obtained from the Heegaard triple  $H$ . There is an epimorphism

$$\pi : H_1(M_{\alpha\beta} \sqcup M_{\beta\gamma} \sqcup M_{\alpha\gamma}, \mathbb{Z}) / \text{Tors} \longrightarrow H_1(W, \mathbb{Z}) / \text{Tors}.$$

Every element  $\zeta \in H_1(W, \mathbb{Z}) / \text{Tors}$  may be represented as  $(\zeta_{\alpha\beta}, \zeta_{\beta\gamma}, \zeta_{\alpha\gamma})$  with

$$(\zeta_{\alpha\beta}, \zeta_{\beta\gamma}, \zeta_{\alpha\gamma}) \in H_1(M_{\alpha\beta} \sqcup M_{\beta\gamma} \sqcup M_{\alpha\gamma}, \mathbb{Z}) / \text{Tors}.$$

We may then define

$$\begin{aligned} \mathfrak{f}_{\alpha\beta\gamma, \mathbf{t}}^{\zeta} &: \text{CF}(\Sigma, \alpha, \beta, \mathbf{u}, \mathbf{t}|_{M_{\alpha\beta}}) \longrightarrow \text{CF}(\Sigma, \alpha, \gamma, \mathbf{u}, \mathbf{t}|_{M_{\alpha\gamma}}) \\ \mathfrak{f}_{\alpha\beta\gamma, \mathbf{t}}^{\zeta}(\mathbf{x}) &:= \mathfrak{f}_{\alpha\beta\gamma, \mathbf{t}}(\zeta_{\alpha\beta} \cdot \mathbf{x} \otimes \Theta_{\beta\gamma} + \mathbf{x} \otimes \zeta_{\beta\gamma} \cdot \Theta_{\beta\gamma}) - \zeta_{\alpha\gamma} \cdot \mathfrak{f}_{\alpha\beta\gamma, \mathbf{t}}(\mathbf{x} \otimes \Theta_{\beta\gamma}). \end{aligned}$$

Correspondingly, we may define

$$\begin{aligned} \mathfrak{f}_{\alpha\beta\gamma, \mathbf{t}}^{\mathbb{M}} &: \text{HF}^{\mathbb{M}}(\mathcal{T}_1) \otimes \Lambda^*(H_1(W, \mathbb{Z}) / \text{Tors}) \longrightarrow \text{HF}^{\mathbb{M}}(\mathcal{T}_2) \\ \mathfrak{f}_{\alpha, \beta, \gamma, \mathbf{t}}^{\mathbb{M}}(x \otimes \zeta) &:= \mathfrak{f}_{\alpha\beta\gamma, \mathbf{t}}^{\zeta}(\mathbf{x}). \end{aligned}$$

Lemma 2.6 from [OS4] then implies that the above map is in fact well-defined, and does not depend on the representation of  $\zeta$  as  $\pi(\zeta_{\alpha\beta}, \zeta_{\beta\gamma}, \zeta_{\alpha\gamma})$ . After composing with the maps  $\mathfrak{f}_{\mathcal{C}_1}^{\mathbb{M}}$  and  $\mathfrak{f}_{\mathcal{C}_2}^{\mathbb{M}}$  we obtain an induced map, which may be denoted by

$$\bar{\mathfrak{f}}_{\mathcal{C}}^{\mathbb{M}} : \text{HF}^{\mathbb{M}}(\mathcal{T}) \otimes \Lambda^*(H_1(W, \mathbb{Z}) / \text{Tors}) \longrightarrow \text{HF}^{\mathbb{M}}(\mathcal{T}').$$

We may then follow the steps taken to show the invariance of the map  $\mathfrak{f}_{\mathcal{C}}^{\mathbb{M}}$  and show that  $\bar{\mathfrak{f}}_{\mathcal{C}}^{\mathbb{M}}$  is also well-defined.

**7.5. Relative  $\text{Spin}^c$  structures and the cobordism map.** Suppose that  $\mathcal{W} = (W, F)$  is a stable cobordism from a balanced tangle  $(M^0, T^0)$  to a balanced tangle  $(M^1, T^1)$ . Let  $(X^i, \tau^i)$  be the sutured manifold associated with  $(M^i, T^i)$  for  $i = 0, 1$ . Thus,  $X^i = M^i - \text{nd}(T^i)$  for a tubular neighborhood  $\text{nd}(T^i)$  around  $T^i$ . Moreover, if  $T^i = \sqcup_{j=1}^{\kappa} T_j^i$  then  $\tau^i = \sqcup_{j=1}^{\kappa} \tau_j^i$  where  $\tau_j^i \subset \partial X^i$  is the meridians of  $T_j^i$  for any  $1 \leq j \leq \kappa$  and  $i = 0, 1$ .

For any  $1 \leq j \leq \kappa$  let  $\chi_j^i \in H^2(X^i, \partial X^i, \mathbb{Z})$  denote the Poincare dual of the homology classes represented by  $\tau_j^i \subset X^i$ . Suppose  $F = \sqcup_{i=1}^m F_i$ . Let

$$\mathcal{I}_F^i := \langle \chi_j^i - \chi_k^i \mid T_j^i, T_k^i \subset F_l \text{ for } 1 \leq l \leq m \rangle \subset H^2(X^i, \partial X^i, \mathbb{Z})$$

for  $i = 0, 1$ . The set of relative  $\text{Spin}^c$  classes on  $(M^i, T^i)$  denoted by  $\text{Spin}^c(M^i, T^i)$  is an affine space on  $H^2(X^i, \partial X^i, \mathbb{Z})$  and thus  $\mathcal{I}_F^i$  acts on it

by translation. Let

$$(6) \quad \begin{aligned} \text{Spin}_F^c(M^i, T^i) &:= \frac{\text{Spin}^c(M^i, T^i)}{\mathcal{I}_F^i} \quad \text{and} \\ \mathbb{H}_F^i &:= \frac{H^2(X^i, \partial X^i, \mathbb{Z})}{\mathcal{I}_F^i} \end{aligned}$$

For any  $1 \leq l \leq m$  let  $\eta_l^i \in \mathbb{H}_F^i$  be the cohomology class determined by  $\chi_j^i$  for a  $T_j^i \subset \partial F_l$ . Then there is an exact sequence

$$0 \longrightarrow \langle \eta_l^i \mid 1 \leq l \leq m \rangle_{\mathbb{Z}} \longrightarrow \text{Spin}_F^c(M^i, T^i) \longrightarrow \text{Spin}^c(M^i) \longrightarrow 0$$

for  $i = 0, 1$ .

Suppose  $\partial^+ M^0 = \bigsqcup_{i=1}^k S_i^-$  and  $\partial^- M^0 = \bigsqcup_{j=1}^l S_j^+$ . Thus

$$\partial_h^- W = \bigsqcup_{i=1}^k S_i^- \times [0, 1] \quad \text{and} \quad \partial_h^+ W = \bigsqcup_{j=1}^l S_j^+ \times [0, 1].$$

Associated with any connected component of  $\partial_h W$ , consider the elements

$$\mathbf{u}_i^- := \prod_{F_j \cap S_i^- \neq \emptyset} \mathbf{u}_j \quad \text{for } 1 \leq i \leq k, \quad \mathbf{u}_i^+ := \prod_{F_j \cap S_i^+ \neq \emptyset} \mathbf{u}_j \quad \text{for } 1 \leq i \leq l$$

in the free  $\mathbb{F}$ -algebra  $\mathbb{F}[\mathbf{u}_1, \dots, \mathbf{u}_m]$  for  $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ . Then,

$$(7) \quad \mathbb{A}_F := \frac{\mathbb{F}[\mathbf{u}_1, \dots, \mathbf{u}_m]}{\langle \mathbf{u}_i^- \mid g(S_i^-) > 0 \rangle + \langle \mathbf{u}_i^+ \mid g(S_i^+) > 0 \rangle + \langle \mathbf{u}^+ - \mathbf{u}^- \rangle}$$

where  $\mathbf{u}^- = \sum_{i=1}^k \mathbf{u}_i^-$  and  $\mathbf{u}^+ = \sum_{i=1}^l \mathbf{u}_i^+$ . The map  $\mathbf{u}_F : \pi_0(F) \rightarrow \mathbb{A}_F$  which maps  $F_i$  to  $\mathbf{u}_i$  for  $i = 1, \dots, m$  gives  $\mathcal{C} = [W, F, \mathbf{t}, \mathbf{u}_F]$  the structure of an  $\mathbb{A}_F$ -cobordism for every  $\mathbf{t} \in \text{Spin}^c(W)$ . Let  $\mathbf{u}^i : \pi_0(T^i) \rightarrow \mathbb{A}_F$  be the map induced by  $\mathbf{u}_F$ .

The algebra  $\mathbb{A}_F$  is filtered by  $\mathbb{H}_F^i$  for  $i = 0, 1$  and the filtration is defined by

$$\begin{aligned} \chi^i : G(\mathbb{A}_F) &\rightarrow \mathbb{H}_F^i = H^2(X^i, \partial X^i, \mathbb{Z}) \\ \chi^i(\prod_{j=1}^m \mathbf{u}_j^{a_j}) &:= \sum_{j=1}^m a_j \eta_j^i. \end{aligned}$$

Therefore, for any  $\text{Spin}^c$  class  $\mathfrak{s}^i \in \text{Spin}^c(M^i)$  we have

$$\text{HF}^{\mathbb{A}_F}(M^i, T^i, \mathfrak{s}^i, \mathbf{u}^i) = \bigoplus_{\underline{\mathfrak{s}}^i \in \mathfrak{s}^i \subset \text{Spin}_F^c(M^i, T^i)} \text{HF}^{\mathbb{A}_F}(M^i, T^i, \underline{\mathfrak{s}}^i, \mathbf{u}^i)$$

**Lemma 7.7.** *Consider the  $\mathbb{A}_F$ -cobordism  $\mathcal{C} = [W, F, \mathbf{t}, \mathbf{u}_F]$  as above and let  $\mathfrak{s}^0 = \mathbf{t}|_{M^0}$  and  $\mathfrak{s}^1 = \mathbf{t}|_{M^1}$ . Suppose that for a generator  $\mathbf{x}$  of  $\text{CF}(M^0, T^0, \mathfrak{s}^0, \mathbf{u}^0)$*

with  $\underline{s}(\mathbf{x}) = \underline{s}^0 \in \mathfrak{s}^0$  we have  $\mathfrak{f}_{\mathcal{C}}(\mathbf{x}) \in \text{CF}(M^1, T^1, \underline{s}^1, \mathbf{u}^1)$ . Then for any generator  $\mathbf{y}$  of  $\text{CF}^{\mathbb{A}_F}(M^0, T^0, \mathfrak{s}^0, \mathbf{u}^0)$  with

$$\underline{s}(\mathbf{y}) - \underline{s}(\mathbf{x}) = \sum_{i=1}^m a_i \eta_i^0$$

we have

$$\mathfrak{f}_{\mathcal{C}}(\mathbf{y}) \in \text{CF}\left(M^1, T^1, \underline{s}^1 + \sum_{i=1}^m a_i \eta_i^1, \mathbf{u}^1\right).$$

**Proof.** It is enough to prove the lemma in the case where  $\mathcal{C} = \mathcal{C}(\mathbb{I}, \mathbb{S}, \mathfrak{t})$  corresponds to a set of framed arcs and a framed link. Let

$$H = (\Sigma, \alpha, \beta, \gamma, \mathbf{z})$$

be a Heegaard triple subordinate to  $(\mathbb{I}, \mathbb{S})$  for  $\mathcal{C}$ . Assume that  $\mathbf{x}$  and  $\mathbf{y}$  are intersection points of  $\mathbb{T}_{\alpha} \cap \mathbb{T}_{\beta}$  such that

$$\underline{s}(\mathbf{y}) - \underline{s}(\mathbf{x}) = \sum_{i=1}^m a_i \eta_i^0.$$

Consider triangle classes  $\Delta_x \in \pi_2(\mathbf{x}, \Theta_{\beta\gamma}, \mathbf{x}')$  and  $\Delta_y \in \pi_2(\mathbf{y}, \Theta_{\beta\gamma}, \mathbf{y}')$  representing the  $\text{Spin}^c$  structure  $\mathfrak{t}$  for  $\mathbf{y}, \mathbf{y}' \in \mathbb{T}_{\alpha} \cap \mathbb{T}_{\gamma}$ . For any  $1 \leq i \leq m$ , let  $n_i(\Delta_{\bullet}) = \sum_{T_j \subset \partial F_i} n_{z_j}(\Delta_{\bullet})$  for  $\bullet = x, y$ . Then

$$\underline{s}\left(\prod_{i=1}^m \mathbf{u}_i^{n_i(\Delta_y)} \cdot \mathbf{y}'\right) - \underline{s}\left(\prod_{i=1}^m \mathbf{u}_i^{n_i(\Delta_x)} \cdot \mathbf{x}'\right) = \sum_{i=1}^m a_i \eta_i^1.$$

This completes the proof of lemma.  $\square$



## 8. APPLICATIONS AND SPECIAL CASES

**8.1. Cobordisms between closed 3-manifolds.** Let  $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$  and  $\mathbb{A} = \mathbb{F}[\mathbf{u}]$ . Consider an oriented closed based 3-manifold  $\mathcal{Y} = (Y, p)$  i.e.  $p \in Y$  is a fixed based point. Associated with  $\mathcal{Y}$  and any given  $\text{Spin}^c$  structure  $\mathfrak{s} \in \text{Spin}^c(Y)$  one may define an  $\mathbb{A}$ -tangle  $\mathcal{T}_{\mathcal{Y}, \mathfrak{s}} = [M_{\mathcal{Y}}, T_{\mathcal{Y}}, \mathfrak{s}, \mathbf{u}_{\mathcal{Y}}]$  as follows. Let  $p_-, p_+ \in Y$  be points close to  $p$  and  $T \subset Y$  be an embedded oriented arc passing through  $p$  such that  $\partial^- T = p_-$  and  $\partial^+ T = p_+$ . Then  $M_{\mathcal{Y}}$  is constructed by removing small disjoint balls (also disjoint from  $p$ ) around  $p_-$  and  $p_+$  and  $T_{\mathcal{Y}} = T \cap M_{\mathcal{Y}}$ . Note that  $\partial^- M_{\mathcal{Y}}$  and  $\partial^+ M_{\mathcal{Y}}$  are the boundary of spheres around  $p_-$  and  $p_+$ , respectively and  $T_{\mathcal{Y}}$  has one connected component. The map  $\mathbf{u}_{\mathcal{Y}} : \pi_0(T_{\mathcal{Y}}) \rightarrow \mathbb{A}$  is defined by  $\mathbf{u}_{\mathcal{Y}}(T_{\mathcal{Y}}) = \mathbf{u}$ .

**Definition 8.1.** Let  $(Y, p)$  and  $(Y', p')$  be oriented closed based 3-manifolds. A decorated cobordism  $\mathcal{X} = (X, \sigma)$  from  $\mathcal{Y} = (Y, p)$  to  $\mathcal{Y}' = (Y', p')$  is a smooth oriented 4-manifold  $X$  with  $\partial X = -Y \sqcup Y'$  and a properly embedded arc  $\sigma \subset X$  such that  $\partial \sigma = p \sqcup p'$ .

Associated with any decorated cobordism  $\mathcal{X} = (X, \sigma)$  from  $\mathcal{Y} = (Y, p)$  to  $\mathcal{Y}' = (Y', p')$  and any  $\text{Spin}^c$  structure  $\mathfrak{t} \in \text{Spin}^c(X)$ , one may construct an  $\mathbb{A}$ -cobordism  $\mathcal{C}_{\mathcal{X}, \mathfrak{t}} = [W_{\mathcal{X}}, F_{\mathcal{X}}, \mathfrak{t}, \mathbf{u}_{\mathcal{X}}]$  from  $\mathcal{T} = \mathcal{T}_{\mathcal{Y}, \mathfrak{s}}$  to  $\mathcal{T}' = \mathcal{T}_{\mathcal{Y}', \mathfrak{s}'}$  as follows. Let  $T \subset Y$  and  $T' \subset Y'$  be embedded oriented arcs containing  $p$  and  $p'$  respectively and let  $\partial^\bullet T = p_\bullet$  and  $\partial^\bullet T' = p'_\bullet$  for  $\bullet = +, -$ . Consider parallel disjoint copies of  $\sigma$  in  $X$ , denoted by  $\sigma^-$  and  $\sigma^+$ , such that  $\partial \sigma^- = p_- \sqcup p'_-$ ,  $\partial \sigma^+ = p_+ \sqcup p'_+$  and  $\sigma^- \cup T' \cup \sigma^+ \cup T$  bound an embedded disk  $D$  in  $X$ . Then  $W_{\mathcal{X}}$  is obtained from  $X$  by removing small disjoint tubular neighborhoods around  $\sigma^-$  and  $\sigma^+$  while  $F_{\mathcal{X}} = D \cap W_{\mathcal{X}}$  and  $\mathbf{u}_{\mathcal{X}}(F_{\mathcal{X}}) = \mathbf{u}$ .

For every closed oriented based 3-manifold  $\mathcal{Y} = (Y, p)$  and every  $\text{Spin}^c$  structure  $\mathfrak{s} \in \text{Spin}^c(Y)$ , the homology groups  $\text{HF}^{\mathbb{M}}(\mathcal{T}_{\mathcal{Y}, \mathfrak{s}})$  for  $\mathbb{M}$  equal to  $\mathbb{F}$ ,  $\mathbb{F}[\mathbf{u}]$ ,  $\mathbb{F}[\frac{1}{\mathbf{u}}]$  or  $\mathbb{F}[\mathbf{u}, \frac{1}{\mathbf{u}}]$  are equal to  $\widehat{\text{HF}}(Y, \mathfrak{s}; \mathbb{F})$ ,  $\text{HF}^-(Y, \mathfrak{s}; \mathbb{F})$ ,  $\text{HF}^+(Y, \mathfrak{s}; \mathbb{F})$  and  $\text{HF}^\infty(Y, \mathfrak{s}; \mathbb{F})$ , respectively. Moreover, for every decorated cobordism  $\mathcal{X} = (X, \sigma)$  from  $(Y, p)$  to  $(Y', p')$  and every  $\text{Spin}^c$  structure  $\mathfrak{t} \in \text{Spin}^c(X)$ , the cobordism map  $\mathfrak{f}_{\mathcal{C}_{\mathcal{X}, \mathfrak{t}}}^{\mathbb{M}}$  corresponding to the associated  $\mathbb{A}$ -cobordism  $\mathcal{C}_{\mathcal{X}, \mathfrak{t}}$  is the cobordism map of Ozsváth and Szabó in any of the aforementioned cases.

**8.2. Functoriality of link Floer homology.** Another important example of  $\mathbb{A}$ -tangles is given by multi-based links.

**Definition 8.2.** A multi-based link is a triple  $\mathcal{L} = (Y, L, \mathbf{p})$  where  $L$  is an oriented link in a closed connected oriented 3-manifold  $Y$  together with a finite set  $\mathbf{p} \subset L$  of based points (or markings) such that any connected component of  $L$  intersects  $\mathbf{p}$ .

Associated with any multi-based link  $\mathcal{L} = (Y, L, \mathbf{p})$  we may define a balanced tangle

$$\mathcal{T}_{\mathcal{L}} = (M_{\mathcal{L}}, T_{\mathcal{L}})$$

as follows. Assume  $\mathbf{p} = \{p_1, \dots, p_n\}$  and  $I = \sqcup_{i=1}^n I_i$  where  $I_i \subset L$  is a connected segment containing  $p_i$ . Considering the orientation induced on  $I$  from  $L$ , let  $\mathbf{p}_- = \partial^- I$  and  $\mathbf{p}_+ = \partial^+ I$ . Then  $M_{\mathcal{L}}$  is obtained from  $Y$  by removing small disjoint ball neighborhoods around the points in  $P_-$  and  $P_+$  and  $\partial^\bullet M_{\mathcal{L}} \subset \partial M_{\mathcal{L}}$  is the union of sphere boundary components corresponding to  $\mathbf{p}_\bullet$  for  $\bullet = +, -$ . Furthermore,

$$T_{\mathcal{L}} = (I \cap M_{\mathcal{L}}) \bigsqcup -((L \setminus I) \cap M_{\mathcal{L}}).$$

**Definition 8.3.** A decorated cobordism from  $\mathcal{L} = (Y, L, \mathbf{p})$  to  $\mathcal{L}' = (Y', L', \mathbf{p}')$  is a triple  $\mathcal{Z} = (Z, F, \sigma)$  as follows.

- (1)  $Z$  is a smooth oriented 4-manifold with  $\partial Z = -Y \sqcup Y'$ .
- (2)  $F \subset Z$  is a properly embedded oriented surface such that  $\partial F = -L \sqcup L'$ .
- (3)  $\sigma \subset F$  is union of properly embedded disjoint oriented arcs such that  $\partial^- \sigma = \mathbf{p}$ ,  $\partial^+ \sigma = \mathbf{p}'$  and if a connected component of  $F - \sigma$  has positive genus then it intersects  $L$  and  $L'$  in more than one connected component.

To any decorated cobordism  $\mathcal{Z}$  from  $\mathcal{L}$  to  $\mathcal{L}'$ , we assign a cobordism  $\mathcal{W}_{\mathcal{Z}} = (W_{\mathcal{Z}}, F_{\mathcal{Z}})$  from the tangle  $\mathcal{T} = \mathcal{T}_{\mathcal{L}}$  to  $\mathcal{T}' = \mathcal{T}_{\mathcal{L}'}$ . We may choose the labeling for  $\mathbf{p} = \sqcup_{i=1}^n p_i$ ,  $\mathbf{p}' = \sqcup_{i=1}^n p'_i$  and  $\sigma = \sqcup_{i=1}^n \sigma_i$  such that  $\partial \sigma_i = p_i \sqcup p'_i$  for any  $1 \leq i \leq n$ . Let  $I = \sqcup_{i=1}^n I_i$  and  $I' = \sqcup_{i=1}^n I'_i$  where  $I_i \subset L$  and  $I'_i \subset L'$  are connected segments containing  $p_i$  and  $p'_i$ , respectively. Let

$$\mathbf{p}_\bullet = \partial^\bullet I = \sqcup_{i=1}^n p_{i\bullet} \quad \text{and} \quad \mathbf{p}'_\bullet = \partial^\bullet I' = \sqcup_{i=1}^n p'_{i\bullet}$$

where  $p_{i\bullet} = \partial^\bullet I_i$  and  $p'_{i\bullet} = \partial^\bullet I'_i$  for  $\bullet = -, +$ . Consider parallel copies  $\sigma_i^-, \sigma_i^+ \subset F$  of each  $\sigma_i$  such that  $\partial \sigma_i^\bullet = p_{i\bullet} \sqcup p'_{i\bullet}$  for  $\bullet = -, +$ . Moreover,  $\sigma_i^- \cup \sigma_i^+ \cup I_i \cup I'_i$  bounds a disk  $D_i \subset F$  containing  $\sigma_i$  such that  $D_1, \dots, D_n$  are pairwise disjoint. Let  $F^\circ = F - (\sqcup_{i=1}^n D_i)$ . Then,  $W_{\mathcal{Z}}$  is constructed from  $Z$  by removing disjoint small tubular neighborhood of the arcs  $\sqcup_{i=1}^n (\sigma_i^- \sqcup \sigma_i^+)$  while

$$F_{\mathcal{Z}} = (\sqcup_{i=1}^n D_i \cap Z_{\mathcal{Z}}) \bigsqcup - (F^\circ \cap Z_{\mathcal{Z}}).$$

Abusing the notation, we denote  $D_i \cap Z_{\mathcal{Z}}$  by  $D_i$  and  $F_j^\circ \cap Z_{\mathcal{Z}}$  by  $F_j^\circ$  where  $F^\circ = \sqcup_{j=1}^m F_j^\circ$ . Let  $\mathbb{A}$  denote the algebra associated with the cobordism  $\mathcal{W}_{\mathcal{Z}}$  as in Equation 7. Thus,

$$\mathbb{A} = \mathbb{F}[\mathbf{v}_1, \dots, \mathbf{v}_m, \mathbf{u}_1, \dots, \mathbf{u}_n]$$

and the map  $\mathbf{u} : \pi_0(F_{\mathcal{Z}}) \rightarrow \mathbb{A}$  defined by

$$\begin{cases} \mathbf{u}(F_j^\circ) = \mathbf{v}_j & 1 \leq j \leq m \\ \mathbf{u}(D_i) = \mathbf{u}_i & 1 \leq i \leq n. \end{cases}$$

gives the triple  $\mathcal{C}_{\mathcal{Z}, \mathbf{t}} = [W_{\mathcal{Z}}, F_{\mathcal{Z}}, \mathbf{t}, \mathbf{u}]$  the structure of an  $\mathbb{A}$ -tangle for every  $\mathbf{t} \in \text{Spin}^c(Z)$ . Let

$$\mathbf{u}_{\mathcal{L}} : \pi_0(T_{\mathcal{L}}) \rightarrow \mathbb{A} \quad \text{and} \quad \mathbf{u}_{\mathcal{L}'} : \pi_0(T_{\mathcal{L}'}) \rightarrow \mathbb{A}$$

denote the maps induced by  $\mathbf{u}$ . Then for every  $\mathbb{A}$ -module  $\mathbb{M}$  we have an  $\mathbb{A}$ -homomorphism

$$\mathfrak{f}_{\mathcal{C}_{\mathcal{Z},\mathfrak{t}}}^{\mathbb{M}} : \mathrm{HF}^{\mathbb{M}}(\mathcal{T}_{\mathcal{L},\mathfrak{s}}) \rightarrow \mathrm{HF}^{\mathbb{M}}(\mathcal{T}_{\mathcal{L}',\mathfrak{s}'})$$

where  $\mathfrak{s} = \mathfrak{t}|_{M_{\mathcal{L}}}$  and  $\mathfrak{s}' = \mathfrak{t}|_{M_{\mathcal{L}'}}$ .

For  $\mathbb{M} = \mathbb{A}$ , following the discussions of Subsection 7.5 the chain complexes  $\mathrm{CF}(\mathcal{T}_{\mathcal{L},\mathfrak{s}})$  and  $\mathrm{CF}(\mathcal{T}_{\mathcal{L}',\mathfrak{s}'})$  are  $(\mathbb{A}, \mathbb{H}_F)$  and  $(\mathbb{A}, \mathbb{H}'_F)$  filtered chain complexes where  $\mathbb{H}_F$  and  $\mathbb{H}'_F$  are defined as in Equation 6. Moreover, the cobordism map  $\mathfrak{f}_{\mathcal{C}_{\mathcal{Z},\mathfrak{t}}}^{\mathbb{A}}$  preserves the relative filtration in the sense of Lemma 7.7. In particular, for  $\mathbb{M} = \mathbb{F}[\mathbf{u}_1, \dots, \mathbf{u}_n]$  which has the structure of an  $\mathbb{A}$ -module via the homomorphism  $\phi : \mathbb{A} \rightarrow \mathbb{M}$  which maps any  $\mathbf{v}_i$  to zero, we obtain

$$\mathfrak{f}_{\mathcal{C}_{\mathcal{Z},\mathfrak{t}}}^{\mathbb{M}} : \mathrm{HFL}^-(Y, L, \mathbf{p}, \mathfrak{s}) \rightarrow \mathrm{HFL}^-(Y', L', \mathbf{p}', \mathfrak{s}').$$

**8.3. Crossing changes and tangle Floer homology.** Let us assume that  $L$  is an oriented null-homologous link inside a 3-manifold  $Y$ . If two points  $p, q \in L$  are connected by a framed arc  $\mathbb{I}$ , we obtain a new link by doing band surgery along  $\mathbb{I}$ . This operation will be called a *resolution*. We will identify the framed arcs  $\mathbb{I}$  with the corresponding bands. We denote the minimum number of resolutions needed for changing a link  $L$  to another link  $L'$  by  $r(L, L')$  and call it the resolution distance between  $L$  and  $L'$ .

Let us now start with an oriented link  $L \subset Y$ , and suppose that  $\mathbb{I} = (\mathbb{I}_1, \dots, \mathbb{I}_m)$  is an ordered sequence of disjoint framed arcs with endpoints on  $L$  so that the result of band surgery on them (i.e. the corresponding resolutions) is another oriented link  $L' \subset Y$ . After doing the resolutions corresponding to  $(\mathbb{I}_1, \dots, \mathbb{I}_i)$  we obtain a knot or a link  $L_i$ . Associated with  $\mathbb{I}$ , we also obtain a surface  $F \subset Y \times [0, m]$ , where

$$F \cap (Y \times \{0\}) = L \times \{0\} \quad \text{and} \quad F \cap (Y \times \{m\}) = L' \times \{m\}.$$

The projection map  $\pi$  from  $Y \times [0, m]$  to the second factor gives a Morse function on  $F$ . We assume that  $F$  is connected. This may be guaranteed if one of  $L$  and  $L'$  is a knot.

We may turn  $L$  into a multi-based link, by choosing a number of markings  $\mathbf{p} = \{p_1, \dots, p_n\}$  on  $L$ . If we further assume that  $\mathbf{p}$  is disjoint from the end points of the bands in  $\mathbb{I}$ , and each connected component of  $L - \mathrm{nd}\{p_1, \dots, p_n\}$  contains at most one connected component of the boundary of  $\mathbb{I}$ , the markings  $\mathbf{p}$  also determine a tangle  $(M, T') = (M_{L',\mathbf{p}}, T_{L',\mathbf{p}})$ . Correspondingly, band surgery on  $\mathbb{I}$  determines an stable cobordism  $(W_{\mathbb{I},\mathbf{p}}, F_{\mathbb{I},\mathbf{p}})$  from  $(M, T)$  to  $(M', T')$ . Let us consider the algebra  $\mathbb{A}_n = \mathbb{F}[\mathbf{u}_1, \dots, \mathbf{u}_n, \mathbf{v}]$  which is freely generated over  $\mathbb{F}$  by  $n+1$  variables and is associated with  $(W_{\mathbb{I},\mathbf{p}}, F_{\mathbb{I},\mathbf{p}})$ . For every  $\mathrm{Spin}^c$  structure  $\mathfrak{s} \in \mathrm{Spin}^c(Y)$  we obtain the  $\mathbb{A}$ -cobordism

$$\mathcal{C} = \mathcal{C}_{\mathbb{I},\mathbf{p},\mathfrak{s}} = [M \times [0, n], F, \mathfrak{s}, \mathbf{u}] : \mathcal{T} = \mathcal{T}_{L,\mathbf{p},\mathfrak{s}} \rightsquigarrow \mathcal{T}' = \mathcal{T}_{L',\mathbf{p},\mathfrak{s}}.$$

Correspondingly, we obtain the chain complexes

$$\mathrm{CF}(L, \mathbf{p}, \mathfrak{s}) = \mathrm{CF}(\mathcal{T}) \quad \text{and} \quad \mathrm{CF}(L', \mathbf{p}, \mathfrak{s}) = \mathrm{CF}(\mathcal{T}')$$

as well as the chain maps

$$\begin{aligned} \mathfrak{f} &= \mathfrak{f}_{\mathbb{I}, \mathbf{p}, \mathfrak{s}} : \mathrm{CF}(L, \mathbf{p}, \mathfrak{s}) \rightarrow \mathrm{CF}(L', \mathbf{p}, \mathfrak{s}) \quad \text{and} \\ \mathfrak{g} &= \mathfrak{g}_{\mathbb{I}, \mathbf{p}, \mathfrak{s}} : \mathrm{CF}(L', \mathbf{p}, \mathfrak{s}) \rightarrow \mathrm{CF}(L, \mathbf{p}, \mathfrak{s}), \end{aligned}$$

where the second map is obtained by reversing the cobordism  $(W, F)$ .

**Lemma 8.4.** *With the above notation fixed, the maps*

$$\mathfrak{f} \circ \mathfrak{g} : \mathrm{CF}(L', \mathbf{p}, \mathfrak{s}) \rightarrow \mathrm{CF}(L, \mathbf{p}, \mathfrak{s}) \quad \text{and} \quad \mathfrak{g} \circ \mathfrak{f} : \mathrm{CF}(L, \mathbf{p}, \mathfrak{s}) \rightarrow \mathrm{CF}(L', \mathbf{p}, \mathfrak{s})$$

*are both multiplication by  $\mathfrak{v}^m$ , up to chain homotopy.*

**Proof.** The  $\mathbb{A}_n$ -cobordism  $\mathcal{C}$  may be decomposed as  $\mathcal{C} = \mathcal{C}_1 \cup_{\mathcal{T}_1} \dots \cup_{\mathcal{T}_{m-1}} \mathcal{C}_m$  where each  $\mathcal{C}_i$  corresponds to a single framed arc  $\mathbb{I}_i$ . It is then enough to show that  $\mathfrak{f}_{\bar{\mathcal{C}}_i} \circ \mathfrak{f}_{\mathcal{C}_i}$  is multiplications by  $\mathfrak{v}$ , where  $\bar{\mathcal{C}}_i$  is obtained by reversing  $\mathcal{C}_i$ . For defining  $\mathfrak{f}_{\mathcal{C}_i}$  we may use a triple

$$(\Sigma, \alpha, \beta, \gamma, \mathbf{z} = \{z_1, \dots, z_{2n}\})$$

subordinate to the framed arc  $\mathbb{I}_i$ , where  $z_i$  is mapped by  $\mathbf{u}$  to  $\mathbf{u}_i$  if  $i = 1, \dots, n$  and to  $\mathfrak{v}$  if  $i > n$ . If  $\delta$  is obtained by a small Hamiltonian isotopy from  $\beta$

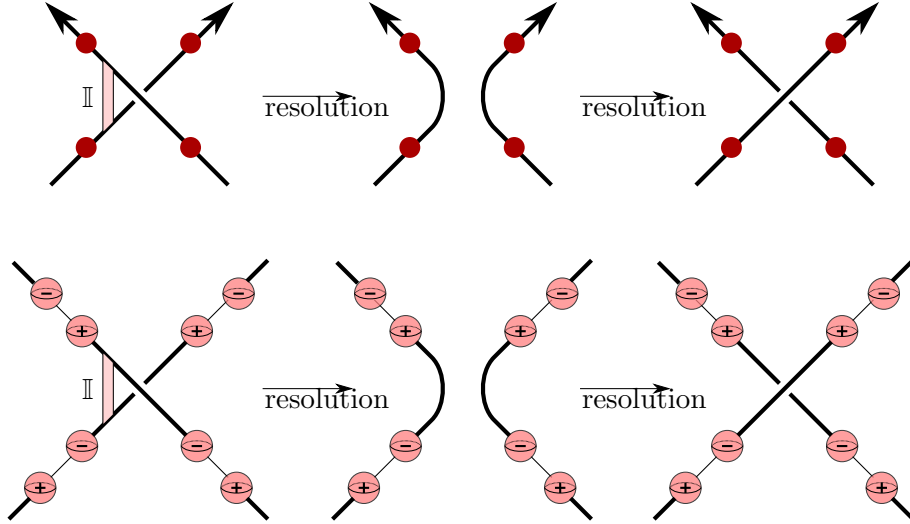


FIGURE 9. A resolution and the corresponding band surgery are illustrated. Corresponding to a marking of the link or knot, we obtain a balanced tangle and corresponding to the framed arcs we obtain a stable cobordism.

which do not cross  $\mathbf{z}$ , then  $(\Sigma, \alpha, \gamma, \delta, \mathbf{z})$  is subordinate to  $\bar{\mathbb{I}}_i$ . Associated with the  $\mathbb{A}$ -diagram

$$H = (\Sigma, \alpha, \beta, \gamma, \delta, u : \mathbf{z} \rightarrow \mathbb{A}_n, \mathfrak{s})$$

is a holomorphic square map  $\mathfrak{S}$  which satisfies

$$d \circ \mathfrak{S} + \mathfrak{S} \circ d = \mathfrak{f}_{\bar{c}_i} \circ \mathfrak{f}_{c_i} + \mathfrak{f}_{\alpha\beta\delta}(- \otimes \mathfrak{f}_{\beta\gamma\delta}(\Theta_{\beta\gamma} \otimes \Theta_{\gamma\delta})).$$

The position of the curves in  $\beta \cup \gamma \cup \delta$ , which is basically illustrated in Figure 10, implies that

$$\mathfrak{f}_{\beta\gamma\delta}(\Theta_{\beta\gamma} \otimes \Theta_{\gamma\delta}) = \mathfrak{v}\Theta_{\beta\delta}.$$

Since  $\mathfrak{f}_{\alpha\beta\delta}(- \otimes \Theta_{\beta\delta})$  gives a map chain homotopic to the identity on  $\text{CF}(L_i, \mathbf{p}, \mathfrak{s})$ , the above observation completes the proof.  $\square$

Let us now assume that  $L' = K$  is a knot and that  $L$  is a link with  $\ell$  components. We may then follow the above process starting with a set  $\mathbf{p} = \{p_1, \dots, p_\ell\}$  of  $\ell$  marked points so that each connected component of  $L$  contains precisely one of  $p_i$ . In this case, we may denote  $\text{CF}(L, \mathbf{p}, \mathfrak{s})$  by  $\text{CF}(L, \mathfrak{s})$  and  $\text{CF}(K, \mathbf{p}, \mathfrak{s})$  by  $\text{CF}(K, \ell, \mathfrak{s})$ , as the position of the markings in  $\mathbf{p}$  is no longer important in the definition. Correspondingly, we obtain the chain maps

$$\mathfrak{f}_{L,K} : \text{CF}(L, \mathfrak{s}) \rightarrow \text{CF}(K, \ell, \mathfrak{s}) \quad \text{and} \quad \mathfrak{g}_{L,K} : \text{CF}(K, \ell, \mathfrak{s}) \rightarrow \text{CF}(L, \mathfrak{s})$$

with  $\mathfrak{f}_{L,K} \circ \mathfrak{g}_{L,K}$  and  $\mathfrak{g}_{L,K} \circ \mathfrak{f}_{L,K}$  both chain homotopic to multiplication by  $\mathfrak{v}^m$ . A direct consequence of the above observation is the following proposition.

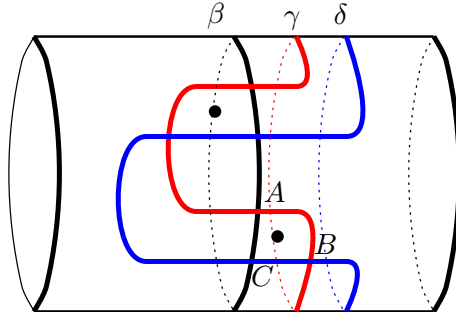


FIGURE 10. If  $\delta$  are obtained from  $\beta$  by Hamiltonian isotopy supported away from the marked points, the domain of the distinguished triangle class in  $(\Sigma, \beta, \gamma, \delta, \mathbf{z})$  which connects the distinguished generators  $\Theta_{\beta\gamma}$ ,  $\Theta_{\gamma\delta}$  and  $\Theta_{\beta\delta}$  will contain one of the marked points corresponding to the strands connected by  $\mathbb{I}$ . The intersection of the domain of the aforementioned triangle with the surface is the small triangle connecting  $A, B$  and  $C$ .

**Proposition 8.5.** *Let us assume that  $K$  is a knot and  $L$  is a link with  $\ell$  components. Let  $m = r'(K, L)$  denote the smallest non-negative integer so that there are chain maps*

$$\mathfrak{f} : \text{CF}(L, \mathfrak{s}) \rightarrow \text{CF}(K, \ell, \mathfrak{s}) \quad \text{and} \quad \mathfrak{g} : \text{CF}(K, \ell, \mathfrak{s}) \rightarrow \text{CF}(L, \mathfrak{s})$$

*with  $\mathfrak{f} \circ \mathfrak{g}$  and  $\mathfrak{g} \circ \mathfrak{f}$  both chain homotopic to multiplication by  $\mathfrak{v}^m$ . Then*

$$r'(K, L) \leq r(K, L).$$

We would now like to apply the above proposition when a particular sequence of resolutions is performed on a knot. By a *crossing change* for an oriented link  $L \subset Y$  we mean replacing a ball in  $Y$  in which  $L$  looks like a positive crossing to the ball in which  $L$  looks like a negative crossing (a positive crossing change), or the reverse of the above operation (a negative crossing change). Figure 11 illustrates how a band surgery on  $L$  can be used to do any of the following two changes:

- Change a positive crossing to a negative crossing and add a component to the link which is a positively oriented meridian for  $L$ , or the reverse of this operation.
- Change a negative crossing to a positive crossing and add a component to the link which is a negatively oriented meridian for  $L$ , or the reverse of this operation.

In particular, if a sequence of crossing changes for a knot  $K$  change it to a new knot  $K'$ , the corresponding band surgeries change  $K$  to a link  $L$  which consists of the knot  $K'$  together with some  $\ell$  copies of its meridian, which are all oriented positively, or all oriented negatively. Here  $\ell$  can be any non-negative integer and is the difference between the numbers of positive and negative crossing changes. We will then write  $L = K'_{\pm\ell}$ . Our construction gives the chain maps

$$\mathfrak{f} : \text{CF}(K, \ell + 1, \mathfrak{s}) \rightarrow \text{CF}(K'_{\pm\ell}, \mathfrak{s}) \quad \text{and} \quad \mathfrak{g} : \text{CF}(K'_{\pm\ell}, \mathfrak{s}) \rightarrow \text{CF}(K, \ell + 1, \mathfrak{s})$$

which induce maps on homology, with  $\mathfrak{f} \circ \mathfrak{g}$  and  $\mathfrak{g} \circ \mathfrak{f}$  both chain homotopic to multiplication by  $\mathfrak{v}^m$ .

Let us assume that  $\mathfrak{u} = \mathfrak{u}_0$  corresponds to  $K'$  and  $\mathfrak{u}_1, \dots, \mathfrak{u}_\ell$  correspond to its meridians, which are all positively oriented. In particular, after this re-labelling of variables, we have  $\mathbb{A}_{\ell+1} = \mathbb{F}[\mathfrak{u}, \mathfrak{u}_1, \dots, \mathfrak{u}_\ell, \mathfrak{v}]$ . There are a chain homotopy equivalences

$$\begin{aligned} \text{CF}(K'_\ell, \mathfrak{s}) &= (\text{CF}(K'_{\ell-1}, \mathfrak{s}) \otimes_{\mathbb{A}_\ell} \mathbb{A}_{\ell+1}) \otimes_{\mathbb{A}_{\ell+1}} W_\ell \quad \text{and} \\ \text{CF}(K, \ell + 1, \mathfrak{s}) &= (\text{CF}(K, \ell, \mathfrak{s}) \otimes_{\mathbb{A}_\ell} \mathbb{A}_{\ell+1}) \otimes_{\mathbb{A}_{\ell+1}} V_\ell \end{aligned}$$

where  $V_\ell$  and  $W_\ell$  is an  $\mathbb{A}_{\ell+1}$ -module freely generated by  $x_1, x_2$  and by  $y_1, y_2, y_3, y_4$ , respectively.  $V_\ell$  is equipped with differential defined by  $d(x_1) = (\mathfrak{u} + \mathfrak{u}_\ell)x_2$  and  $d(x_2) = 0$ , while  $W_\ell$  is equipped with the differential

$$d(y_1) = (\mathfrak{u} + \mathfrak{u}_\ell)y_4, \quad d(y_2) = \mathfrak{v}y_3 + \mathfrak{u}_\ell y_4 \quad \text{and} \quad d(y_3) = d(y_4) = 0.$$

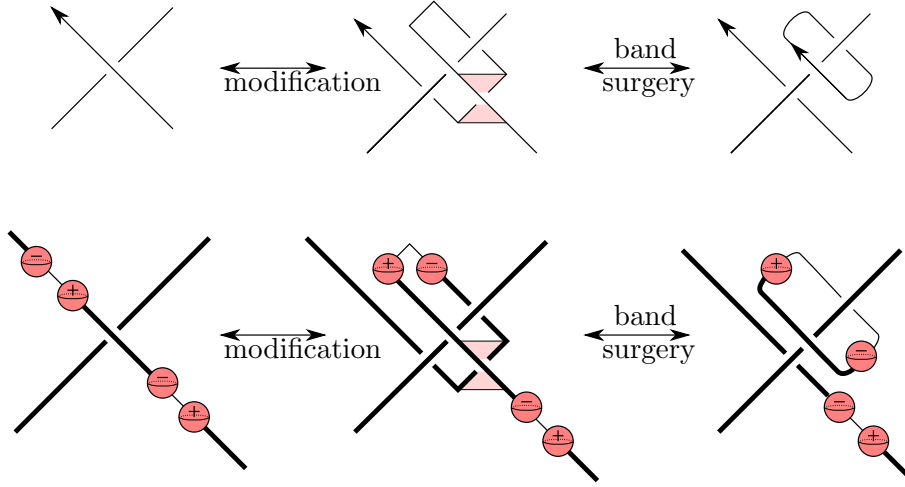


FIGURE 11. We may change a crossing in the expense of adding a meridian. The meridian can be positively or negatively oriented depending on whether the initial crossing is negatively or positively oriented, respectively.

Corresponding to the above two chain homotopy equivalences, we may present  $\mathfrak{f}$  and  $\mathfrak{g}$  as  $(\mathfrak{f}_{ij})$  and  $(\mathfrak{g}_{ji})$  with  $i = 1, \dots, 4$  and  $j = 1, 2$ , where

$$\begin{aligned} \mathfrak{f}_{ij} &: \text{CF}(K, \ell, \mathfrak{s}) \otimes_{\mathbb{A}_\ell} \mathbb{A}_{\ell+1} \rightarrow \text{CF}(K'_{\ell-1}, \mathfrak{s}) \otimes_{\mathbb{A}_\ell} \mathbb{A}_{\ell+1} \quad \text{and} \\ \mathfrak{g}_{ji} &: \text{CF}(K_{\ell-1}, \mathfrak{s}) \otimes_{\mathbb{A}_\ell} \mathbb{A}_{\ell+1} \rightarrow \text{CF}(K, \ell, \mathfrak{s}) \otimes_{\mathbb{A}_\ell} \mathbb{A}_{\ell+1}. \end{aligned}$$

Let us denote  $\mathbf{u} + \mathbf{u}_\ell$  by  $\sigma_\ell$ . The equations  $d \circ \mathfrak{f} = \mathfrak{f} \circ d$  and  $d \circ \mathfrak{g} = \mathfrak{g} \circ d$  each give 8 equations among  $\{\mathfrak{f}_{ij}\}_{i,j}$  and  $\{\mathfrak{g}_{ji}\}_{i,j}$ . In particular, we obtain:

$$\begin{aligned} \sigma_\ell \mathfrak{f}_{12} &= d \circ \mathfrak{f}_{11} + \mathfrak{f}_{11} \circ d, & \sigma_\ell \mathfrak{f}_{22} &= d \circ \mathfrak{f}_{12} + \mathfrak{f}_{12} \circ d \\ \sigma_\ell \mathfrak{g}_{13} &= d \circ \mathfrak{g}_{23} + \mathfrak{g}_{23} \circ d \quad \text{and} \quad \sigma_\ell \mathfrak{g}_{14} &= d \circ \mathfrak{g}_{24} + \mathfrak{g}_{24} \circ d. \end{aligned}$$

We may regard the ring  $\mathbb{A}_{\ell+1}$  as  $\mathbb{A}_\ell[\sigma_\ell]$ . The differential  $d$  of the complexes does not use this new variable  $\sigma_\ell$ . On the other hand, we may write

$$\mathfrak{f}_{ij} = \sum_{k=0}^{\infty} \mathfrak{f}_{ij,k} \sigma_\ell^k \quad \text{and} \quad \mathfrak{g}_{ij} = \sum_{k=0}^{\infty} \mathfrak{g}_{ij,g} \sigma_\ell^k.$$

Let us define  $\mathfrak{f}'_{ij} = \sum_{k=1}^{\infty} \mathfrak{f}_{ij,k} \sigma_\ell^{k-1}$  and  $\mathfrak{g}'_{ij} = \sum_{k=1}^{\infty} \mathfrak{g}_{ij,k} \sigma_\ell^{k-1}$ . The above equations then imply

$$\begin{aligned} \mathfrak{f}_{12} &= d \circ \mathfrak{f}'_{11} + \mathfrak{f}'_{11} \circ d, & \mathfrak{f}_{22} &= d \circ \mathfrak{f}'_{12} + \mathfrak{f}'_{12} \circ d \\ \mathfrak{g}_{13} &= d \circ \mathfrak{g}'_{23} + \mathfrak{g}'_{23} \circ d \quad \text{and} \quad \mathfrak{g}_{14} &= d \circ \mathfrak{g}'_{24} + \mathfrak{g}'_{24} \circ d. \end{aligned}$$

Let  $P^f$  denote the  $4 \times 2$  matrix with  $P_{i2}^f = \mathfrak{f}'_{i1}$  and  $P_{i1}^f = 0$  and  $P^g$  denote the  $2 \times 4$  matrix with  $P_{1i}^g = \mathfrak{g}'_{2i}$  and  $P_{2i}^g = 0$ . We may then replace  $\mathfrak{f}$  by  $\mathfrak{f} + (P^f \circ d + d \circ P^f)$  and replace  $\mathfrak{g}$  by  $\mathfrak{g} + (P^g \circ d + d \circ P^g)$ , which are chain homotopic with the initial maps. Let us denote the new matrices by  $\mathfrak{f}^0 = (\mathfrak{f}_{ij}^0)$

and  $\mathbf{g}^0 = (\mathbf{g}_{ij}^0)$ . It is then easy to check that  $\mathbf{f}_{12}^0 = \mathbf{f}_{22}^0 = \mathbf{g}_{13}^0 = \mathbf{g}_{14}^0 = 0$ , while  $\mathbf{f}_{ij}^0 = \mathbf{f}_{ij,0}$  and  $\mathbf{g}_{ji}^0 = \mathbf{g}_{ji,0}$  for other values of  $(i, j)$ . As a result,

$$\mathbf{f}_{ij}^0 : \text{CF}(K, \ell, \mathfrak{s}) \rightarrow \text{CF}(K'_{\ell-1}, \mathfrak{s}) \quad \text{and} \quad \mathbf{g}_{ji}^0 : \text{CF}(K'_{\ell-1}, \mathfrak{s}) \rightarrow \text{CF}(K, \ell, \mathfrak{s})$$

We can now set  $\sigma_\ell = 0$ , or equivalently  $\mathbf{u}_\ell = \mathbf{u}$ . Since  $\mathbf{f}^0$  and  $\mathbf{g}^0$  are both chain maps we find that  $\mathbf{f}_{ij}^0$  is a chain map except for  $(i, j) \in \{(3, 1), (4, 1)\}$  and  $\mathbf{g}_{ji}^0$  is also a chain map except for  $(i, j) = (2, 2)$ . Furthermore

$$\begin{aligned} \mathbf{v}\mathbf{f}_{21}^0 &= d \circ \mathbf{f}_{31}^0 + \mathbf{f}_{31}^0 \circ d, & \mathbf{u}\mathbf{f}_{21}^0 &= d \circ \mathbf{f}_{41}^0 + \mathbf{f}_{41}^0 \circ d \\ \text{and} \quad \mathbf{v}\mathbf{g}_{23}^0 + \mathbf{u}\mathbf{g}_{24}^0 &= d \circ \mathbf{g}_{22}^0 + \mathbf{g}_{22}^0 \circ d. \end{aligned}$$

Let us define

$$\begin{aligned} \mathbf{f}_\ell &= \mathbf{f}_{11}^0 : \text{CF}(K, \ell, \mathfrak{s}) \rightarrow \text{CF}(K'_{\ell-1}, \mathfrak{s}) \quad \text{and} \\ \mathbf{g}_\ell &= \mathbf{g}_{11}^0 : \text{CF}(K'_{\ell-1}, \mathfrak{s}) \rightarrow \text{CF}(K, \ell, \mathfrak{s}). \end{aligned}$$

We can then compute

$$\begin{aligned} (\mathbf{f}^0 \circ \mathbf{g}^0)_{11} &= \mathbf{f}_{11}^0 \circ \mathbf{g}_{11}^0 + \mathbf{f}_{12}^0 \circ \mathbf{g}_{21}^0 = \mathbf{f}_\ell \circ \mathbf{g}_\ell \quad \text{and} \\ (\mathbf{g}^0 \circ \mathbf{f}^0)_{11} &= \sum_{i=1}^4 \mathbf{g}_{1i}^0 \circ \mathbf{f}_{i1}^0 = \mathbf{g}_\ell \circ \mathbf{f}_\ell + \mathbf{g}_{12}^0 \mathbf{f}_{21}^0. \end{aligned}$$

On the other hand we have

$$\begin{aligned} \mathbf{v}\mathbf{g}_{12}^0 \mathbf{f}_{21}^0 &= d \circ (\mathbf{g}_{12}^0 \mathbf{f}_{31}^0) + (\mathbf{g}_{12}^0 \mathbf{f}_{31}^0) \circ d \quad \text{and} \\ \mathbf{u}\mathbf{g}_{12}^0 \mathbf{f}_{21}^0 &= d \circ (\mathbf{g}_{12}^0 \mathbf{f}_{41}^0) + (\mathbf{g}_{12}^0 \mathbf{f}_{41}^0) \circ d. \end{aligned}$$

It follows from the above two observations that  $\mathbf{f}_\ell \circ \mathbf{g}_\ell$  is chain homotopic to multiplication by  $\mathbf{v}$ , while  $\mathbf{u}\mathbf{g}_\ell \circ \mathbf{f}_\ell$  and  $\mathbf{v}\mathbf{g}_\ell \circ \mathbf{f}_\ell$  are chain homotopic to  $\mathbf{u}\mathbf{v}$  and  $\mathbf{v}^2$ , respectively.

If we repeat the above argument for the rest of meridians, we find the chain maps

$$\bar{\mathbf{f}} : \text{CF}(K, \mathfrak{s}) \rightarrow \text{CF}(K', \mathfrak{s}) \quad \text{and} \quad \bar{\mathbf{g}} : \text{CF}(K', \mathfrak{s}) \rightarrow \text{CF}(K, \mathfrak{s})$$

with  $\bar{\mathbf{f}} \circ \bar{\mathbf{g}}$  chain homotopic to multiplication by  $\mathbf{v}^m$  and  $\mathbf{u}' \bar{\mathbf{g}} \circ \bar{\mathbf{f}}$  chain homotopic to multiplication by  $\mathbf{u}' \mathbf{v}^m$  for every monomial  $\mathbf{u}'$  of degree  $\ell$  in  $\mathbb{A} = \mathbb{F}[\mathbf{u}, \mathbf{v}]$ .

Note that we may instead assume that  $\mathbf{g} \circ \mathbf{f}$  is chain homotopic to multiplication by  $\mathbf{v}^m$ , while  $\mathbf{u}' \mathbf{f} \circ \mathbf{g}$  is multiplication by  $\mathbf{u}' \mathbf{v}^m$ . This could have been done by leaving the meridians on  $K$  and doing the stabilization on  $K'$ . However, the condition that  $\mathbf{u}' \mathbf{f} \circ \mathbf{g} : \mathbb{A} \rightarrow \mathbb{A}$  is chain homotopic to  $\mathbf{u}' \mathbf{v}^m$  means that  $\mathbf{f} \circ \mathbf{g}$  is multiplication by  $\mathbf{v}^m$ , since  $\mathbb{F}[\mathbf{u}, \mathbf{v}]$  is a division ring.

A similar argument may be applied when we are left with a number of negatively oriented meridians, and we will only need to make some small modifications. Alternatively, we may change the orientation of the ambient



manifold to change all negative crossings to positive and all positive crossings to negative.

Note that for knots  $K \subset S^3$ ,  $\text{CF}(K)$  is filtered by relative  $\text{Spin}^c$  structures, which live in  $\underline{\text{Spin}}^c(S^3, K) = \mathbb{Z}$ . This filtration is given by a map  $\underline{\mathfrak{s}}$  from the set of generators to  $\mathbb{Z}$ , and by setting

$$\underline{\mathfrak{s}}(\mathbf{u}^a \mathbf{v}^b \mathbf{x}) = \underline{\mathfrak{s}}(\mathbf{x}) - a + b \in \mathbb{Z},$$

for every generator  $\mathbf{x}$  and every pair  $(a, b)$  of non-negative integers. Subsequently, we may write

$$\text{CF}(K) = \bigoplus_{s \in \mathbb{Z}} \text{CF}(K, s).$$

In particular, for the unknot we obtain

$$\mathbb{A} = \mathbb{F}[\mathbf{u}, \mathbf{v}] = \bigoplus_{s \in \mathbb{Z}} \mathbb{A}(s), \quad \text{where } \mathbb{A}(s) = \langle \mathbf{u}^a \mathbf{v}^b \mid b - a = s \rangle.$$

The chain maps  $\bar{\mathfrak{f}}$  and  $\bar{\mathfrak{g}}$  which are constructed above respect the relative  $\text{Spin}^c$  structures, by Lemma 7.7. In particular, there are integers  $m_f$  and  $m_g$  with  $m_f + m_g = m$  such that

$$\mathfrak{f}(\text{CF}(K, s)) \subset \mathbb{A}(s + m_f) \quad \text{and} \quad \mathfrak{g}(\text{CF}(K, s)) \subset \mathbb{A}(s + m_g).$$

It is implied from the proof of Lemma 8.4 that both  $m_f$  and  $m_g$  are non-negative. We call such chain maps homogeneous chain maps of degrees  $m_f$  and  $m_g$ , respectively.

**Definition 8.6.** Let  $\mathbb{A} = \mathbb{F}[\mathbf{u}, \mathbf{v}]$ . For a knot  $K \subset S^3$ , let  $u'(K)$  denote the least integer  $m$  with the property that there are homogeneous chain maps

$$\mathfrak{f} : \text{CF}(K) \rightarrow \mathbb{A} \quad \text{and} \quad \mathfrak{g} : \mathbb{A} \rightarrow \text{CF}(K)$$

of non-negative degrees  $m_f$  and  $m_g$  such that  $\mathfrak{f} \circ \mathfrak{g} = \mathbf{v}^m$  while the composition

$$\mathfrak{g} \circ \mathfrak{f} : \text{CF}(K) \rightarrow \text{CF}(K)$$

is chain homotopic to multiplication by  $\mathbf{v}^m$ .

The outcome of the above discussion is the following theorem.

**Theorem 8.7.** If  $K$  is a knot in  $S^3$ , then  $u'(K)$  is a lower bound for the unknotting number  $u(K)$ .

Unlike some of the other bounds from Heegaard Floer theory for the unknotting number of knots in  $S^3$  which are also lower bounds for the slice genus,  $u'(K)$  does not bound the slice genus from below. In fact, if  $K$  is non-trivial then  $u'(K)$  is positive. Nevertheless,  $u'(K)$  is always greater than or equal to  $\tau(K)$ .

**Proposition 8.8.** For every knot  $K \subset S^3$ ,  $\tau(K) \leq u'(K)$ .

**Proof.** Let us assume that there are homogeneous chain maps

$$\mathfrak{f} : \text{CF}(K, \mathbb{A}) \rightarrow \mathbb{A} \quad \text{and} \quad \mathfrak{g} : \mathbb{A} \rightarrow \text{CF}(K, \mathbb{A})$$

of non-negative degrees  $m_f$  and  $m_g$ , satisfying the condition of Definition 8.6. If we use the  $\mathbb{A}$ -module  $\mathbb{M} = \mathbb{F}[\mathfrak{v}]$  for computing the homology in the above theorem, for every non-negative integer  $s \in \mathbb{Z}$  which represents a relative  $\text{Spin}^c$  class on  $(S^3, K)$  we obtain the maps

$$\mathfrak{g}_s : \mathbb{M}(s) \rightarrow \text{HF}^{\mathbb{M}}(K, s + m_g) = H_*(\text{CF}(K, s + m_g) \otimes_{\mathbb{A}} \mathbb{M}).$$

The vector space  $\mathbb{M}(s)$  is trivial for  $s < 0$  and is generated by  $\mathfrak{v}^s$  for  $s \geq 0$ . If  $s \geq 0$  then  $\mathfrak{f}_{s+m_g} \circ \mathfrak{g}_s$  is the identity map from  $\mathbb{M}(s)$  to  $\mathbb{M}(s + m)$ . On the other hand, note that the chain complexes  $\{\text{CF}(K, s) \otimes_{\mathbb{A}} \mathbb{M}\}_{s \in \mathbb{Z}}$  give the standard filtration of the knot chain complex which is used in the definition of  $\tau(K)$ . Thus  $\text{HF}^{\mathbb{M}}(K, s)$  is naturally mapped to  $\mathbb{F}$ , and the map is non-trivial if and only if  $s \geq \tau(K)$ . On the other hand, if we set  $\mathfrak{v} = 1$  we obtain a new module  $\mathbb{M}'$ , and  $\text{HF}^{\mathbb{M}'}(K) = \widehat{\text{HF}}(S^3) = \mathbb{F}$ , implying that both  $\mathfrak{f}^{\mathbb{M}'}$  and  $\mathfrak{g}^{\mathbb{M}'}$  are isomorphisms on  $\mathbb{F}$ , i.e. both are the identity. This implies that the map from  $\text{HF}^{\mathbb{M}}(K, s)$  to  $\mathbb{F}$  given by the filtration, which was discussed above, is precisely  $\mathfrak{f}_s$ . In particular, the map  $\mathfrak{f}_s$  is non-trivial if and only if  $s \geq \tau(K)$ .

Since  $\mathfrak{f}_{s+m_g} \circ \mathfrak{g}_s$  is the identity map from  $\mathbb{M}(s) = \mathbb{F}$  to  $\mathbb{M}(s + m) = \mathbb{F}$  for every  $s \geq 0$ , we conclude that  $m_g \geq \tau(K)$ . This completes the proof.  $\square$

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